The Stability of Plane Poiseuille Flow

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The problem of the stability of plane Poiseuille flow to small disturbances leads to a characteristic value problem for the Orr-Sommerfeld equation with given boundary conditions. It happens that negative values of the imaginary parts of the characteristic numbers, which indicate instability, are small, at any rate over the region here investigated, and considerable accuracy is required to establish them, while the Reynolds numbers for which they occur are large.

In this paper the fourth-order differential equation is replaced by a difference system of the same order with a truncation error involving the eighth derivative, so that the error is sufficiently small with a reasonably large interval. The resulting system of linear algebraic equations is solved by direct Gaussian elimination, which avoids the difficulties due to rapid exponential growth of error for high Reynolds number which beset the standard integration procedure.

The characteristic value is obtained for a range of Reynolds numbers and wavelengths of the disturbance, and the critical Reynolds number found to be 5780 for wavelength 3.062 times the width of the channel. A detailed discussion of the accuracy of the work is given for the (unstable) case of wavelength π and Reynolds number 10 000, and a table of the profile of the disturbance is given for this case.

I. PLANE DISTURBANCE OF PLANE POISEUILLE FLOW

THE equations of motion of a homogeneous incompressible viscous fluid in two dimensions under hydrostatic pressure between two planes at $y=\pm b$ are satisfied, to the first order in ϵ by the stream function

$$\psi=U_0\left(\frac{y^3}{3b^2}-y\right)+\epsilon\{\exp[-i\alpha(x-ct)]\varphi(y)+\exp[i\alpha(x-\bar{c}t)]\bar{\varphi}(y)\},$$

where c and φ are complex and $\bar{c}$ and $\bar{\varphi}$ their complex conjugates, provided that the profile of the disturbance $\varphi(y)$ satisfies the Orr-Sommerfeld equation,

$$d^4\varphi/dy^4-2\alpha^2\ddot{\varphi}+\alpha^4\varphi+\frac{i\alpha U_0}{\nu}\times\left\{\left(1-\frac{c}{U_0}-\frac{y}{b^2}\right)(\ddot{\varphi}-\alpha^2\varphi)+\frac{2}{b^2}\varphi\right\}=0,$$

and the boundary conditions $\varphi=0$, $\dot{\varphi}=0$, at $y=\pm b$.

U_0 is the speed of the undisturbed flow in the x direction at the center of the stream; $2\pi/\alpha$ is the wavelength of the disturbance in the x direction, and c is the complex velocity of the disturbance, which is damped so long as the imaginary part of c is positive. ν is the kinematic viscosity of the fluid. Dots signify differentiation with respect to y.

Writing $R=U_0b/\nu$, the Reynolds' number, and replacing y by by, x by bx, α by α/b, t by bt/U_0, c by cU_0, ψ by $bU_0\psi$, and φ by $bU_0\varphi$, we have the reduced form

$$d^4\varphi/dy^4-2\alpha^2\ddot{\varphi}+\alpha^4\varphi+i\alpha R\{(1-c-y^2)(\ddot{\varphi}-\alpha^2\varphi)+2\varphi\}=0.$$

With boundary conditions $\varphi=0$, $\dot{\varphi}=0$ at $y=\pm1$, and corresponding stream function,[1]

$$\psi=(\tfrac{1}{3}y^3-y)+\epsilon\{\exp[-i\alpha(x-ct)]\varphi(y)+\exp[i\alpha(x-\bar{c}t)]\bar{\varphi}(y)\}.$$

Solutions of the differential equation for φ, for given α and R, can be made to satisfy the boundary conditions only for characteristic values of c. We are interested in determining whether there are real values of R and α for which c has a negative imaginary part. The corresponding disturbance then grows in amplitude exponentially with the time. We also wish to determine the critical Reynolds' number, the lowest value of R for which instability of steady motion exists.

Work on this problem by many authors using asymptotic series, culminating in that of Heisenberg and Lin,[2] indicates that the flow becomes unstable for $\alpha=1$ at about $R=5300$, but this value is such that the asymptotic series are not really accurate, and the negative values reached by the imaginary part of c are small. It has, therefore, seemed desirable to attack the problem by direct numerical solution of the equation for assigned real values of α and R and complex values of c, to vary c till the boundary conditions can be satisfied, and then, varying α and R, to determine the minimum value of R for which the characteristic value c has negative imaginary part. The difficulty of this direct solution lies in the large values, for the values of R required, of the coefficients of terms other than that containing the highest-differential coefficient, in the differential equation.

II. REPLACING THE DIFFERENTIAL EQUATION BY A DIFFERENCE EQUATION

In the differential equation,

$$d^4\varphi/dy^4+Pd^2\varphi/dy^2+Q\varphi=0,$$

we may replace the differential coefficients by their expressions in terms of central differences at interval w,

[1] S. Goldstein, *Modern Developments in Fluid Dynamics* (Clarendon Press, Oxford, 1938), p. 197.

[2] See C. C. Lin, Quart. Appl. Math. 3, 287 (1946), where a long list of references to earlier work is given.

$$\frac{1}{w^4}\left(\delta^2-\frac{1}{12}\delta^4+\frac{1}{90}\delta^6-\frac{1}{560}\delta^8\cdots\right)^2\varphi$$
$$+P\frac{1}{w^2}\left(\delta^2-\frac{1}{12}\delta^4+\frac{1}{90}\delta^6-\frac{1}{560}\delta^8\cdots\right)\varphi+Q\varphi=0.$$

In this equation we should normally take terms up to a definite order in w^2, counting δ^2 of order w^2: the first terms neglected then estimate the truncation error. We may, however, without extra difficulty, take terms of one higher order in w^2 in $Pd^2\varphi/dy^2$, since this gives terms of the same order in δ^2. This offsets a large coefficient in P, so far as the truncation error is concerned, when w is small enough.

The truncation error may be made of higher order by substituting

$$\varphi=(1+k_1\delta^2+k_2\delta^4+\cdots)g$$

in the difference equation, and choosing k_1, k_2, $\cdots$ to make the coefficients of the first terms previously neglected vanish.[3] We may also allow k_1, $k_2\cdots$ to contain P and Q and may operate on the equation with $1+l_1\delta^2+l_2\delta^4+\cdots$, introducing also differences of P and Q, obtaining still greater accuracy with a difference equation of given order at the expense of more complicated coefficients. A still more accurate difference equation may be built up from approximate local solutions of the differential equation.

If, in particular, we take

$$g=\varphi-\tfrac{1}{6}w^2\ddot{\varphi}+(w^4/90)d^4\varphi/dy^4,$$

we obtain

$$\varphi=g+\frac{1}{6}\delta^2g+\frac{1}{360}\delta^4g-\left[\frac{67w^8}{907200}\varphi^{(8)}\right],$$
$$\ddot{\varphi}=\frac{1}{w^2}\left(\delta^2g+\frac{1}{12}\delta^4g\right)-\left[\frac{w^6}{6048}\varphi^{(8)}\right],$$
$$d^4\varphi/dy^4=\frac{1}{w^4}\delta^4g-\left[\frac{w^4}{240}\varphi^{(8)}\right].$$

TABLE I. Values of C_0' for various values of c, in case $\alpha=1$, $R=10\,000$, $w=0.02$.[a]

c	$10^{11}C_0'$	First divided difference	Second divided difference
$0.235\ -0.0055i$	$13174731-24320802i$		
		$1232678+8458436i$	
$0.235\ -0.005i$	$8945463-23704463i$		$74469+49110i$
		$1159013+8570140i$	
$0.235\ -0.004i$	$375323-22545450i$		
		$1146092+8672570i$	
$0.2355-0.004i$	$948369-18209165i$		$79653+50376i$
		$1345224+8798509i$	
$0.2375-0.004i$	$3638818-612147i$		
		$1489720+8930980i$	
$0.2375-0.0036i$	$66426-16259i$		$87416+50233i$
		$1459580+9013430i$	
$0.2375-0.0034i$	$-1736261+275657i$		

[a] The interpolated value $c=0.237500592853-0.003592509400i$ gave $10^{11}C_0'=66-196i$, and the value $c=0.2375006-0.0035925i$ was adopted.

[3] This is the Gauss-Jackson-Noumerov method. B. V. Noumerov, Monthly Notices Roy. Astron. Soc. **84**, 592 (1924); J. Jackson, Monthly Notices Roy. Astron. Soc. **84**, 602 (1924).

TABLE II. Values of c for various numbers of steps, in case $\alpha=1$, $R=10\,000$.

Number of steps	c
25	$0.2376559-0.0016981i$
50	$0.2375006-0.0035925i$
75	$0.2375196-0.0037115i$
100	$0.2375243-0.0037312i$
extrapolated to ∞	$0.2375259-0.0037404i$

The terms in brackets, taken at some point in the relevant interval, give the truncation error.

Further we find

$$\dot{\varphi}=\frac{1}{w}\mu\delta g+\left[\frac{w^4}{120}\varphi^{(5)}\right],$$

the term in $\mu\delta^3$ having zero coefficient.

The Orr-Sommerfeld equation for the Poiseuille flow then becomes, retaining only the largest error terms,

$$\delta^4g\left[\left(\frac{1}{w^4}-\frac{2\alpha^2}{12w^2}+\frac{\alpha^4}{360}\right)+i\alpha R\left\{(1-c-y^2)\right.\right.$$
$$\left.\left.\times\left(\frac{1}{12w^2}-\frac{\alpha^2}{360}\right)+\frac{2}{360}\right\}\right]+\delta^2g\left[\left(-\frac{2\alpha^2}{w^2}+\frac{\alpha^4}{6}\right)\right.$$
$$\left.+i\alpha R\left\{(1-c-y^2)\left(\frac{1}{w^2}-\frac{\alpha^2}{6}\right)+\frac{2}{6}\right\}\right]$$
$$+g[\alpha^4+i\alpha R\{(1-c-y^2)(-\alpha^2)+2\}]$$
$$+\left[\left\{\frac{w^4}{240}-i\alpha R(1-c-y^2)\frac{w^6}{6048}\right\}\varphi^{(8)}\right]=0,$$

with the following boundary conditions:

$$g+\frac{1}{6}\delta^2g+\frac{1}{360}\delta^4g-\left[\frac{67w^8}{907200}\varphi^{(8)}\right]=0,$$
$$\frac{1}{w}\mu\delta g+\left[\frac{w^4}{120}\varphi^{(5)}\right]=0,$$

at $y=\pm1$. We shall deal only with even solutions over range $0\leq y\leq 1$, which must satisfy $\mu\delta g=0$, $\mu\delta^3g=0$ at $y=0$.

We see that for $w=0.01$, the second term in the error of the differential equation does not exceed the first till $\alpha R(1-c-y^2)=250\,000$, but that for $w=0.04$, it will exceed it when $\alpha R(1-c-y^2)=16\,000$.

We expect a proportional error per step behaving like $(w^4/100)(\dot{\varphi}/\varphi)^4$, so for $w=0.01$, we might ordinarily expect a truncation error of order 10^{-10}, but if there are regions where $\dot{\varphi}/\varphi$ is large, for example, near $y=1$, where we expect $\dot{\varphi}/\varphi\sim(\iota\alpha Rc)^{\frac{1}{2}}$, we have proportional error $\alpha^2R^2c^2w^4/200$, and for $R=10\,000$, $c=0.25$, $\alpha=1$, this gives 0.0003. However, this will give a total error multiplied by the effective amplitude φ in this neighborhood, and the final accuracy can only be determined *a posteriori*.

TABLE III. Values of C_0' for various values of c, in case $\alpha=1$, $R=10\,000$, $w=0.005$.

c	$10^{11}C_0'$
$2.38 -0.040i$	$44760+62816i$
$2.376-0.040i$	$37228+5138i$
$2.375-0.036i$	$-19357+1340i$
$2.375259-0.037404i$	$-1555-1063i$
$2.37524808332-0.03731799983i$	$-2077-828i$
$2.37524994756-0.03732369406i$	$1027-1829i$
$2.37524303556-0.03731157406i$	$-2170+1864i$

Note that for a difference equation as above of the same order as the differential equation it approximates, we may expect the same number of linearly independent solutions with the same kind of behavior as for the differential equation.

III. SOLUTION OF THE DIFFERENCE SYSTEM

The resulting system of difference equations has the form, for 100 steps,

$$\begin{aligned}
C_0\, g_0 + D_0\, g_1 + E_0\, g_2 &= 0,\\
B_1\, g_0 + C_1\, g_1 + D_1\, g_2 + E_1\, g_3 &= 0,\\
A_2\, g_0 + B_2\, g_1 + C_2\, g_2 + D_2\, g_3 + E_2\, g_4 &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\\
A_{98}\, g_{96} + B_{98}\, g_{97} + C_{98}\, g_{98} + D_{98} g_{99} + E_{98} g_{100} &= 0,\\
A_{99}\, g_{97} + B_{99}\, g_{98} + C_{99}\, g_{99} + D_{99} g_{100} &= 0,\\
A_{100} g_{98} + B_{100} g_{99} + C_{100} g_{100} &= 0,
\end{aligned}$$

where all the coefficients are linear in the complex characteristic number c.

If we try to solve these by assuming the ratio of g_1 to g_0 and finding in succession g_2, g_3, $\cdots$, we fail for any large Reynolds number unless a very large number of digits is carried, because one solution of the differential equation has a logarithmic rate of increase like $[i\alpha R(1-c-y^2)]^{\frac{1}{2}}$, so that for $R=10\,000$, about 50 decimal digits would be needed to obtain a 5- or 6-digit accuracy in the linearly independent solution which does increase rapidly.

If, however, we solve by direct Gaussian elimination, or any equivalent method, obtaining in succession

$$\begin{aligned}
A_{99}' g_{97} + B_{99}' g_{98} + C_{99}' g_{99} &= 0,\\
A_{98}' g_{96} + B_{98}' g_{97} + C_{98}' g_{98} &= 0,\\
\cdots\cdots\cdots\cdots\cdots &\\
A_2'\, g_0 + B_2'\, g_1 + C_2'\, g_2 &= 0,\\
B_1'\, g_0 + C_1'\, g_1 &= 0,\\
C_0'\, g_0 &= 0,
\end{aligned}$$

and if c has been chosen so that $C_0'=0$, a solution exists, and we can find in succession the ratios of g_1, g_2, $\cdots g_{100}$ to g_0.

This method would give the solution of inhomogeneous equations as if the Green's function of the differential system were used, and we shall run into no difficulty unless one or more of the numbers C_{99}', C_{98}', $\cdots C_1'$, used as divisors, is small.

The first attempt at solution by this method, made in the opposite direction, led to almost random variation of C_{100}' as c was altered near the characteristic value. This is explained by the boundary condition $\varphi=0$, $\dot{\varphi}=0$, at $y=1$, requiring g_{99} and g_{100} to be small, and so C_{98}', in this solution, to be small. A small change of c would then make C_{97}' small, and C_{100}' would be very sensitive to changes in c. Eliminating towards g_0, however, leads to no difficulty, only the final C_0' being small.

The value $c=0.3231+0.0262i$ for $\alpha=1.0$, $R=1600$, which had been established by previous numerical work under the direction of Von Neumann, Pekeris, and Lin, using a method devised by Von Neumann, was used as a starting point and test of the setup.[4] An extrapolated value of c was assumed for neighboring values of α and R, and C_0' computed. Values of C_0' were also obtained for neighboring values of c, and inverse interpolation was used to get a value of c giving $C_0'\approx 0$.

IV. THE RESULTS OF THE COMPUTATION AND THEIR ACCURACY

The computing was done on the International Business Machine Corporation's Selective Sequence Electronic Computer by Donald A. Quarles, Jr., and Phyllis K. Brown. The single accuracy multiplication on this machine is 14×14 decimal digits. The coefficients of the difference equation were scaled to be less than unity and carried to 11 decimals, allowing 2 digits to the left of the decimal point for overflow and one further place for checking by using two different posi-

TABLE IV. Characteristic values of c.

α/R	1600	2500	6400	10 000	35 000
0.9		$0.2857+0.0212i$	$0.2444+0.0012i$	$0.2261-0.0040i$	
1.0	$0.3231+0.0262i$	$0.3011+0.0142i$	$0.2569-0.0009i$	$0.2375-0.0037i$	$0.1886+0.0009i$
1.1	$0.3384+0.0206i$	$0.3148+0.0108i$	$0.2677+0.0007i$	$0.2470+0.0003i$	$0.1911+0.0116i$
1.2		$0.3267+0.0170i$	$0.2763+0.0056i$	$0.2535+0.0075i$	

For $\alpha=1.05$, $R=8000$, $c=0.2524-0.0017i$.
For $\alpha=1.026$, $R=5780$, a 50-step solution gave $\alpha Rc10^{-3}=1.56886522464+0.00043540544i$; a 100-step solution gave $\alpha Rc10^{-3}=1.56899069990-0.00000341519i$.
Solutions were also found for $\alpha=1.025$ and $\alpha=1.027$ for $R=5780$.

[4] This work, also done on International Business Machines Corporation's Selective Sequence Electronic Computer in 1950, was not published as it did not settle the question at issue.

TABLE V.

y	$\varphi(y)$	$\varphi(y)$	for $\alpha=1$, $R=10\,000$	y	$\varphi(y)$
0.00	1.000000+0.000000*i*	0.34	0.903749+0.000753*i*	0.68	0.580157+0.003160*i*
0.01	0.999919+0.000001*i*	0.35	0.897844+0.000799*i*	0.69	0.566035+0.003255*i*
0.02	0.999675+0.000003*i*	0.36	0.891748+0.000846*i*	0.70	0.551578+0.003346*i*
0.03	0.999270+0.000006*i*	0.37	0.885459+0.000895*i*	0.71	0.536773+0.003430*i*
0.04	0.998701+0.000010*i*	0.38	0.878976+0.000945*i*	0.72	0.521606+0.003504*i*
0.05	0.997970+0.000016*i*	0.39	0.872296+0.000997*i*	0.73	0.506058+0.003563*i*
0.06	0.997076+0.000023*i*	0.40	0.865417+0.001050*i*	0.74	0.490106+0.003605*i*
0.07	0.996020+0.000032*i*	0.41	0.858338+0.001104*i*	0.75	0.473721+0.003630*i*
0.08	0.994799+0.000041*i*	0.42	0.851056+0.001160*i*	0.76	0.456871+0.003645*i*
0.09	0.993416+0.000052*i*	0.43	0.843569+0.001218*i*	0.77	0.439520+0.003663*i*
0.10	0.991868+0.000064*i*	0.44	0.835873+0.001276*i*	0.78	0.421630+0.003706*i*
0.11	0.990155+0.000078*i*	0.45	0.827968+0.001337*i*	0.79	0.403168+0.003810*i*
0.12	0.988278+0.000092*i*	0.46	0.819849+0.001399*i*	0.80	0.384105+0.004017*i*
0.13	0.986235+0.000109*i*	0.47	0.811515+0.001462*i*	0.81	0.364427+0.004381*i*
0.14	0.984027+0.000126*i*	0.48	0.802963+0.001527*i*	0.82	0.344137+0.004958*i*
0.15	0.981652+0.000145*i*	0.49	0.794188+0.001594*i*	0.83	0.323259+0.005800*i*
0.16	0.979109+0.000165*i*	0.50	0.785190+0.001662*i*	0.84	0.301837+0.006949*i*
0.17	0.976399+0.000186*i*	0.51	0.775963+0.001732*i*	0.85	0.279937+0.008425*i*
0.18	0.973521+0.000208*i*	0.52	0.766505+0.001803*i*	0.86	0.257644+0.010218*i*
0.19	0.970473+0.000232*i*	0.53	0.756812+0.001876*i*	0.87	0.235055+0.012282*i*
0.20	0.967255+0.000258*i*	0.54	0.746880+0.001951*i*	0.88	0.212274+0.014527*i*
0.21	0.963866+0.000284*i*	0.55	0.736706+0.002027*i*	0.89	0.189408+0.016819*i*
0.22	0.960304+0.000312*i*	0.56	0.726285+0.002104*i*	0.90	0.166567+0.018982*i*
0.23	0.956570+0.000341*i*	0.57	0.715614+0.002183*i*	0.91	0.143870+0.020800*i*
0.24	0.952662+0.000372*i*	0.58	0.704686+0.002264*i*	0.92	0.121459+0.022032*i*
0.25	0.948579+0.000404*i*	0.59	0.693499+0.002346*i*	0.93	0.099519+0.022424*i*
0.26	0.944320+0.000437*i*	0.60	0.682046+0.002430*i*	0.94	0.078317+0.021734*i*
0.27	0.939884+0.000472*i*	0.61	0.670322+0.002516*i*	0.95	0.058235+0.019744*i*
0.28	0.935268+0.000508*i*	0.62	0.658322+0.002603*i*	0.96	0.039820+0.016475*i*
0.29	0.930473+0.000545*i*	0.63	0.646039+0.002692*i*	0.97	0.023815+0.011990*i*
0.30	0.925497+0.000584*i*	0.64	0.633467+0.002782*i*	0.98	0.011157+0.006851*i*
0.31	0.920338+0.000624*i*	0.65	0.620600+0.002874*i*	0.99	0.002900+0.002186*i*
0.32	0.914995+0.000666*i*	0.66	0.607431+0.002969*i*	1.00	0.000000+0.000000*i*
0.33	0.909466+0.000709*i*	0.67	0.593953+0.003064*i*		

tions in the multiplier. The successive coefficients remained between unity and one-tenth in most of the integrations.

The values of C_0' for various values of c, in the case $\alpha=1$, $R=10\,000$, for a 50-step solution, as well as first divided differences and some second divided differences are given in Table I. We see that the values vary regularly except for the right-hand three digits, and, in fact, interpolation led to the last value of c given for which C_0' has only the right-hand three digits. We would perhaps not expect to lose as much accuracy in 50 steps, but no particular care was taken to maintain maximum accuracy per step. We thus obtain

$$c=0.2375006-0.0035925i$$

with an error of about $\frac{1}{5}$ in the last place retained.

The values of c for 25-step, 50-step, 75-step, and 100-step solutions for this case are given in Table II. The last three of these show differences consistent with a truncation error of order w^4, and extrapolation leads to a value

$$c=0.2375259-0.0037404i,$$

of which the right-hand two digits may be uncertain.

Some 200-step solutions were done for values of c near the above, Table III, but C_0' could be reduced only to four digits, and c was not determined more accurately.

Results for various values of α and R are given in Table IV. Only four decimals have been retained because in many cases only 50-step solutions were made. The values are believed accurate to 0.5 in the last place.

Interpolation gives a critical Reynolds number $R=5780$ for $\alpha=1.026$, and integration leads to a value of c with imaginary part close to zero for this value of R and α, but while R is well determined, α is hardly determined better than within the interval 1.02 to 1.03.

These numbers confirm Lin's results closely, and it may now be regarded as proved that plane Poiseuille flow becomes unstable at about $R=5800$. It may be noted that for a given value of α, the flow is unstable only for a finite range range of Reynold's numbers as was also found by Lin and Heisenberg.[2]

Assuming $g_0=1$, we find $g_1, g_2, \cdots, g_{100}$ in succession. $\varphi_0, \varphi_1, \cdots, \varphi_{99}$ are then found from $g+\frac{1}{6}\delta^2 g+(1/360)\delta^4 g$, noting that $g_{-1}=g_1$, $g_{-2}=g_2$, $g_{101}=g_{99}$, while $\varphi_{100}=0$. The results for $\alpha=1$, $R=10\,000$ are given in Table V. They were computed to nine decimals and rounded to six decimals after adjusting φ_0 to unity. The values are very smooth and the truncation error estimated from the differences is at most 1 in the last place retained and approaches this value only near $y=1$.

The variation of φ for other values of α and R for which solutions were obtained is equally smooth and shows no signs of oscillation or other peculiarity than a more rapid change of phase near the boundary.