On the stability of heterogeneous shear flows. Part 2

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Small disturbances relative to a horizontally stratified shear flow are considered on the assumptions that the velocity and density gradients in the undisturbed flow are non-negative and possess analytic continuations into a complex velocity plane. It is shown that the existence of a singular neutral mode (for which the wave speed is equal to the mean speed at some point in the flow) implies the existence of a contiguous, unstable mode in a wave-number (α), Richardson-number (J) plane. Explicit results are obtained for the rate of growth of nearly neutral disturbances relative to Hølmboe's shear flow, in which the velocity and the logarithm of the density are proportional to tanh (y/h). The neutral curve for this configuration, $J = J_0(\alpha)$, is shown to be single-valued. Finally, it is shown that a relatively simple generalization of Hølmboe's density profile leads to a configuration having multiple-valued neutral curves, such that increasing J may be destabilizing for some range(s) of α.

1. Introduction

We shall consider here the stability of a parallel shear flow $U(y)$ in a horizontally stratified, perfect, incompressible fluid of density $\rho(y)$ extending from $y = y_1$ to $y = y_2$. Defining

$$\lambda(y) = \log[\rho(0)/\rho(y)], \tag{1.1}$$

we shall impose the *a priori* restrictions that the vorticity $U'(y)$ and the static stability $\lambda'(y)$ be positive-definite functions of y in the open interval (y_1, y_2) that may be continued analytically into the complex-y plane in the neighbourhood of (y_1, y_2). We also shall neglect the inertial effects of density stratification (Boussinesq approximation), an approximation tantamount to the restriction

$$\lambda'(y)\,h \ll 1, \tag{1.2}$$

where h is an appropriate characteristic length (which we need not fix at this stage).

Following Drazin & Howard (1961), we choose a Cartesian (x, y) co-ordinate system moving in the positive-x direction with the average of the velocities at $y = y_1$ and $y = y_2$ and measure y from the plane of this average velocity. Then, using the subscripts 1 and 2 to imply evaluation at $y = y_1$ and $y = y_2$, we choose

$$V = U_2 = -U_1 \quad \text{and} \quad \sigma = \tfrac{1}{2}(\lambda_2 - \lambda_1) \tag{1.3a, b}$$

as characteristic measures of speed and density change and define

$$J = \sigma g h / V^2 \quad \text{and} \quad \alpha = kh \tag{1.4a, b}$$

as the (reference value of the) Richardson number and the dimensionless wave number of a periodic disturbance of wavelength $2\pi/k$. We also define

$$J_l(y) = g\lambda'(y)/U'^2(y) \tag{1.5}$$

as the local Richardson number.

Having these definitions, we may recapitulate the following theorems with respect to small disturbances of the basic flow.

(i) The stream function† for an (infinitesimal) unstable wave-like disturbance must be of the form

$$\psi(y)\, e^{ik(x-ct)}, \quad k > 0, \quad c = c_r + ic_i \tag{1.6}$$

with $c_i > 0$.

(ii) The complex wave speed for any unstable mode must lie inside the semicircle based on (U_1, U_2).

(iii) A stability boundary (or neutral curve) consists of singular neutral modes (SNM's)—i.e. modes for which $c_i = 0$ and $U(y_c) = c_r$ with $y_1 < y_c < y_2$.

(iv) The stream function for an SNM must be of the form

$$\psi = (y-y_c)^{\frac{1}{2}(1+\nu)} f(y) \quad (y_c < y \leqslant y_2) \tag{1.7a}$$

$$= (y_c - y)^{\frac{1}{2}(1+\nu)} f(y)\, e^{-\frac{1}{2}(1+\nu)\, i\pi} \quad (y_1 \leqslant y < y_c), \tag{1.7b}$$

where $f(y)$ is an analytic function of y in the neighbourhood of (y_1, y_2) that may be taken to be real (by factoring a complex constant, if necessary) for y in $[y_1, y_2]$. The parameter ν is given by

$$\nu = [1 - 4J_l(y_c)]^{\frac{1}{2}}, \tag{1.8}$$

and must be real in (1.7).

(v) A sufficient condition for stability is $J_l(y) > \frac{1}{4}$ everywhere in (y_1, y_2).

(vi) The neutral curve for an infinite shear flow ($y_2 = -y_1 = \infty$) is given by $J = \alpha$ in the joint limit $J, \alpha \to 0$. If $0 < J < \alpha \ll 1$ there is one and only one unstable mode for given α and J; moreover, the principle of exchange of stabilities holds in this neighbourhood in the sense that $c_r/c_i \to 0$ as $c_i \to 0+$.

Theorems (i), (iii), (iv) and (v) were proved by the writer (Miles 1961, hereinafter designated as I), (ii) and (v) by Howard (1961), and (vi) by Drazin & Howard (1961). Theorem (vi) depends on the approximation (1.2), but Theorems (i)–(v) do not. Theorems (iv) and (vi) require $U'(y)$ to be positive definite in (y_1, y_2), but Theorems (i)–(iii) and (v) do not.‡

Theorem (vi) implies the existence of at least one neutral curve and a contiguous unstable mode for an important class of shear flows. This neutral curve is not necessarily unique (see below), however, and it therefore is of interest to establish the existence of unstable modes in the neighbourhood of any neutral curve for non-small α and/or J. We shall achieve this goal in §§ 2–4 below through a general formulation that permits the explicit calculation of an unstable mode

† Subsequently, we shall refer to $\psi(y)$ simply (if loosely) as the stream function.

‡ Howard proved that $g\lambda'(y) - \frac{1}{4}U'^2(y) \geqslant 0$ in (y_1, y_2) is sufficient for stability provided that $U(y)$ is continuous and piecewise twice continuously differentiable. The proof of Theorem (v) given by Miles (1961) was based on the additional restriction $U'(y) > 0$ in (y_1, y_2). Drazin & Howard (1961) do not state all of (vi) above, but it may be inferred from (18) in their paper after posing the restriction $U'(y) > 0$ in (y_1, y_2).

by iteration from its antecedant SNM. Underlying this formulation is the fundamental result that:

(vii) The characteristic equation for the eigenvalue problem may be placed in the form $F(\alpha, J, c) = 0$, where F is an entire function of each of α and J and an analytic function of c in any complex domain that excludes the domain of U, *qua* independent variable for ψ. We may choose the c-domain as the semi-circle of Theorem (ii), which excludes the end points U_1 and U_2, and the U-domain as an arc that connects U_1 and U_2 and lies outside of this semi-circle.

A direct corollary is that:

(viii) If an eigensolution exists for some set of α, J and c, say (α_0, J_0, c_0), then c is a continuous function of α and J in the neighbourhood of (α_0, J_0, c_0); accordingly, the existence of a neutral curve in an (α, J)-plane implies the existence of contiguous, complex eigenvalues; conversely, the (α, J)-trajectory of a complex eigenvalue with a positive imaginary part can terminate only on a neutral curve.

Theorem (vii) is due essentially to Lin (1945; see especially pp. 222, 223), who established it for homogeneous shear flows ($J = 0$); the extension of his proof to our problem is straightforward. It is based on the facts that the coefficients of the linear differential equation for ψ, (2.8) below, are entire functions of each of α and J and analytic functions of c in the aforementioned domains and that the boundary conditions on ψ are independent of (or, more generally, they could be entire functions of) α, J and c.† (We note that the function X, introduced in place of F in §4 below, is not an entire function of J in consequence of the change of dependent variable from ψ to χ; however, X is an entire function of ν, the parameter introduced in place of J in this transformation.)

Theorem (viii) follows from an expansion of F about (α_0, J_0, c_0), together with the remark that if (α_0, J_0) is a point on a neutral curve, the value of c corresponding to a neighbouring point can be real only if this point also lies on the neutral curve. We emphasize that c is not generally an analytic function of α and J; in particular, $c(\alpha, J)$ may have algebraic branch points, implied by $(\partial F/\partial c)_0 = 0$ (e.g. $\alpha_0 = J_0 = c_0 = 0$ in §5 below). We also emphasize that the expansion of F may not converge uniformly with respect to $|c_0 - U_1|$ or $|c_0 - U_2|$. The end points $U_{1,2}$ are excluded as possible eigenvalues for c if $J > 0$, but special difficulties could arise for $\alpha_0 \to 0$, $J_0 \to 0$, $c_0 \to U_{1,2}$ (cf. Lin's discussion of homogeneous shear flows).

We shall illustrate this general formulation by considering the velocity profile

$$U(y) = V \tanh(y/h) \tag{1.9}$$

in conjunction with the density profile

$$\lambda(y) = [1 - r\,\mathrm{sech}^2(y/h)] \tanh(y/h), \quad -\tfrac{1}{2} \leqslant r < 1. \tag{1.10}$$

† Lin (1945) was interested in proving the existence of *unstable* eigenvalues in the neighbourhood of an eigenvalue $c = c_s = U(y_s)$, where y_s is a flex point for the velocity profile. No such restriction is implied on c in Theorem (vii); on the other hand, Theorem (viii) does not guarantee the existence of unstable eigenvalues, although their existence is implied by the development of §4 below.

The special cases $r = 0$ and $r = 1$ have been considered previously by Hølmboe (1960) and Garcia (1961), respectively, but they started from the assumption $c = 0$ and did not prove that their results comprised all possible SNM's. We shall demonstrate that: only stationary ($c = 0$) SNM's are admissible if $r \leqslant 0{\cdot}895$; the most critical SNM's are stationary, and hence the principle of exchange of stabilities holds, for $r \leqslant 0{\cdot}947$; at least two nonstationary SNM's exist for $r_* < r < 1$, where $0{\cdot}956 < r_* < 1$.

Perhaps the most interesting property of the configuration of (1.9) and (1.10) is that (§ 6 below) the neutral curve $J = J_0(\alpha)$ is not single-valued if $r > \frac{1}{2}$ and consists of more than one distinct branch if $r > 0{\cdot}895$. Previous examples of multi-valued neutral curves for heterogeneous shear flows are known, to be sure, but these generally have involved domains in which $J_l(y)$ was arbitrarily small.† For example, Goldstein (1931) considered

$$\frac{U(y)}{V} = \begin{Bmatrix} 1 \\ y \\ -1 \end{Bmatrix} \quad \text{and} \quad \lambda(y) = \begin{Bmatrix} \sigma \\ 0 \\ -\sigma \end{Bmatrix} \quad \text{in} \quad \begin{Bmatrix} y > h \\ -h < y < h \\ y < -h \end{Bmatrix} \tag{1.11}$$

and obtained the two neutral curves (in the present notation)

$$J = 2\alpha(1 \mp e^{-2\alpha})^{-1} - 1. \tag{1.12}$$

These are qualitatively similar to the lowest branches obtained by Garcia (1961; see (6.10a) below with $r = 1$ and $n = 0$ therein).

2. General formulation

The boundary-value problem for $\psi(y)$ on the basis of the approximation (1.2) is given by (see, e.g., I)

$$\psi'' + [g\lambda'(U-c)^{-2} - U''(U-c)^{-1} - k^2]\,\psi = 0 \tag{2.1}$$

and
$$\psi_1 = \psi_2 = 0, \tag{2.2}$$

together with the auxiliary condition that the path of integration must pass below the branch point at $y = y_c$ (cf. (1.7) and (4.11)).

We shall consider ψ as a function of the dimensionless velocity

$$z = U(y)/V, \tag{2.3}$$

and introduce the functions $S(z)$ and $B(z)$ through the transformations

$$U'(y) = (V/h)\,S(z) \tag{2.4}$$

and
$$\lambda(y) = \sigma \int_0^z B(z)\,dz. \tag{2.5}$$

These definitions imply $z_1 = -1$, $z_2 = 1$, and

$$\lambda'(y) = (\sigma/h)\,B(z)\,S(z). \tag{2.6}$$

† We also recall that there are configurations for which no neutral curve exists. Such is the case if $U'(y)$ and $\lambda'(y)$ are constant above a solid boundary (see Taylor 1931 and Eliassen, Høiland & Riis, 1953).

We also introduce the dimensionless wave speed

$$z_c = c/V, \tag{2.7}$$

and note that $|z_c| < 1$ for unstable modes in consequence of Howard's 'semicircle' theorem ((ii) above). Substituting (2.3)–(2.7) into (2.1) and (2.2), we obtain

$$\frac{d}{dz}\left[S(z)\frac{d\psi}{dz}\right]+\left[\frac{JB(z)}{(z-z_c)^2}-\frac{S'(z)}{z-z_c}-\frac{\alpha^2}{S(z)}\right]\psi = 0, \tag{2.8}$$

and

$$\psi = 0 \quad (z = \pm 1). \tag{2.9}$$

One of the advantages of the differential equation (2.8), compared with (2.1), is that it allows us to place rather immediate constraints on those configurations that will admit of reasonably simple analysis. Let us suppose, for example, that $S(z)$ and $B(z)$ are polynomials, that $S(z)$ has at least two, and only, simple zeros, and that the degree of $B(z)$ does not exceed that of $S(z)$. Then (2.8) is a differential equation of the Fuchsian type (Ince 1944, p. 370)—although not the most general of that type—and we may represent its solution by the Riemann symbol

$$\psi = P\begin{Bmatrix} z_c & z_1 & \cdots & z_M & \infty & \\ \tfrac{1}{2}(1+\nu) & \mu_1 & \cdots & \mu_M & \tfrac{1}{2}(M-1+\tau) & z \\ \tfrac{1}{2}(1-\nu) & -\mu_1 & \cdots & -\mu_M & \tfrac{1}{2}(M-1-\tau) & \end{Bmatrix}, \tag{2.10}$$

where

$$S(z_m) = 0, \quad \mu_m = -\alpha/S'(z_m) \quad (m = 1, \ldots, M), \tag{2.11}$$

$$\tau = \left[(M+1)^2 - 4J\lim_{z\to\infty}\frac{B(z)}{S(z)}\right]^{\frac{1}{2}}, \tag{2.12}$$

and (cf. (1.8))

$$\nu = (1-4J_c)^{\frac{1}{2}}, \quad J_c = JB_c/S_c. \tag{2.13}$$

The subscript c implies evaluation at $z = z_c$.

3. Singular neutral modes

We may pose the solution to (2.8) for an SNM in the form

$$\psi = (z-z_c)^{\frac{1}{2}(1+\nu)}g(z;\alpha)\,\phi(z;z_c,\alpha,\nu), \tag{3.1}$$

where we have factored $(z-z_c)^{\frac{1}{2}(1+\nu)}$ and $g(z;\alpha)$ in order to render ϕ an analytic function of z in the neighbourhoods of $z = z_c$ and $z = \pm 1$, respectively. We may achieve the latter goal and satisfy the boundary conditions (2.9) either by choosing

$$g \equiv 1 \quad \text{and} \quad \phi(\pm 1; z_c, \alpha, \nu) = 0 \quad \text{if} \quad S(\pm 1) > 0 \tag{3.2a}$$

or by requiring g to be an analytic function of z in the neighbourhood of $(-1, 1)$ that vanishes at $z = \pm 1$ according to

$$g(z;\alpha) \sim (1 \mp z)^{\mu_\pm} \quad \text{if} \quad S(\pm 1) = 0, \tag{3.2b}$$

where $\mu_\pm$ are defined by (2.11) with $z = \pm 1$. The notation $g(z;\alpha)$ implies that g is to be independent of both z_c and ν (which would not be possible if B were permitted to have poles at $z = \pm 1$).

It follows from (3.1) and (3.2), together with Theorems (iii) and (iv), that $\phi(z; z_c, \alpha, \nu)$ must be an analytic function of z in the neighbourhood of $[-1, 1]$ that may be normalized to be real in $[-1, 1]$.

Substituting (3.1) into (2.8), we may place the result in the form

$$L\phi = (p\phi')' + (z-z_c)^{-1} S^{-1} pq\phi = 0, \tag{3.3}$$

where
$$p = (z-z_c)^{1+\nu} g^2(z;\,\alpha)\, S(z), \tag{3.4}$$

$$q = q(z;\,z_c,\alpha,\nu) = \frac{(1-\nu^2)(S_c B - B_c S)}{(4B_c)(z-z_c)} + \frac{(1+\nu)g'S}{g} - \tfrac{1}{2}(1-\nu)S' + \left[\frac{(Sg')'}{g} - \frac{\alpha^2}{S}\right](z-z_c), \tag{3.5}$$

and the primes imply differentiation with respect to z. We observe that q is an analytic function of z in the neighbourhood of $[-1,1]$ and, *qua* function of the independent parameters z_c, α and ν (rather than z_c, α and J), an analytic function of z_c in the neighbourhood of $[-1,1]$ and an entire function of each of α and ν.

We shall be especially concerned with configurations for which both $S(z)$ and $B(z)$ are even functions of z (U and λ odd functions of y). It then follows from (3.3) and (3.5), together with the boundary conditions on ϕ and the fact that ϕ must be regular at $z = z_c$, that q and ϕ have the reciprocal properties (for real z and z_c)

$$q(-z;\,-z_c,\alpha,\nu) = -q(z;\,z_c,\alpha,\nu) \tag{3.6}$$

and
$$\phi(-z;\,-z_c,\alpha,\nu) = \phi(z;\,z_c,\alpha,\nu). \tag{3.7}$$

A convenient choice of $g(z)$ for such configurations is

$$g(z) = S^{\mu}(z), \quad \mu = -\alpha/S_1', \tag{3.8}$$

where $S_1' \equiv S'(z)$, $z = 1$. Substituting (3.8) into (3.5), we obtain

$$\begin{aligned} q(z,z_c) = {} & \tfrac{1}{4}(1-\nu^2)\,[(S_c/B_c)\,B - S]\,(z-z_c)^{-1} \\ & + [(1+\nu)\,\mu - \tfrac{1}{2}(1-\nu)]\,S' + \mu(z-z_c)\,S'' \\ & + \alpha^2[(S'/S_1')^2 - 1]\,S^{-1}(z-z_c). \end{aligned} \tag{3.9}$$

4. Nearly neutral modes

Now let us suppose that an SNM exists for $\nu = \nu_0(\alpha)$ and $z_c = z_0(\alpha)$, say

$$\phi_0(z) \equiv \phi(z;\,z_0,\alpha,\nu_0), \tag{4.1}$$

and seek a contiguous (α fixed, $\nu \to \nu_0$ and $z_c \to z_0$) solution to (2.8) in the form

$$\psi = (z-z_c)^{\frac{1}{2}(1+\nu)} g(z;\,\alpha)\,\phi_0(z)\,\chi(z;\,z_c,\alpha,\nu). \tag{4.2}$$

Having satisfied the boundary conditions (2.9) in accordance with either (3.2*a*) or (3.2*b*), we require χ to be regular at $z = \pm 1$. On the other hand, χ must be singular at $z = z_c$ if (4.2) is to represent an unstable mode, for ψ must contain the solution of exponent $\frac{1}{2}(1-\nu)$, as well as that of $\frac{1}{2}(1+\nu)$, in this neighbourhood if there is to be a transfer of energy from mean flow to disturbance (see I, § 5). We also note that χ must have poles at the zeros, if any, of ϕ_0; but this has no substantial effect on the following analysis.

Substituting (4.2) into (2.8), we may place the result in the form

$$(p\phi_0^2\chi')' + (\phi_0 L\phi_0)\,\chi = 0, \tag{4.3}$$

where the operator L is defined by (3.3)–(3.5); but $L\phi_0 \neq 0$ unless $z_c = z_0$ and $\nu = \nu_0$. Invoking the condition

$$L_0\phi_0 = 0 \quad (z_c = z_0,\ \nu = \nu_0), \tag{4.4}$$

we obtain

$$\phi_0 L\phi_0 = \phi_0\left[L\phi_0 - \left(\frac{z-z_c}{z-z_0}\right)^{\nu} L_0\phi_0\right] \tag{4.5a}$$

$$= (z-z_c)^{\nu} g^2(z;\alpha) R(z; z_c, \alpha, \nu), \tag{4.5b}$$

where

$$R = -(z_c - z_0) S\phi_0\phi_0'' + [(\nu-\nu_0) S - (z_c - z_0) g^{-2}(g^2 S)']\phi_0\phi_0' + (q-q_0)\phi_0^2, \tag{4.6}$$

and

$$q_0 \equiv q(z; z_0, \alpha, \nu_0). \tag{4.7}$$

Referring to the properties of q and ϕ established in the preceding section, we remark that R is an analytic function of z in the neighbourhood of $[-1, 1]$, an analytic function of z_c in the neighbourhood of $z_c = z_0$, and an entire function of ν that tends uniformly to zero as $\nu \to \nu_0(\alpha)$ and $z_c \to z_0(\alpha)$. We therefore may construct the required solution to (4.3) in the neighbourhood of $z_c = z_0$ and $\nu = \nu_0$ through a joint expansion of R in $z_c - z_0$ and $\nu - \nu_0$.

We may proceed to such a construction through the integral equation†

$$\chi = 1 - \int_1^z \frac{d\xi}{p(\xi)\phi_0^2(\xi)} \int_1^\xi (\eta - z_c)^{\nu} g^2(\eta) R(\eta)\chi(\eta)\, d\eta, \tag{4.8}$$

which follows from two integrations of (4.3) and the normalization

$$\chi(1) = 1. \tag{4.9}$$

The requirement that χ be regular at $z = -1$ then yields

$$X(z_c, \alpha, \nu) = \int_{-1}^{1} (z-z_c)^{\nu} g^2(z) R(z)\chi(z)\, dz = 0 \tag{4.10}$$

as the eigenvalue equation. (We also may obtain (4.10) simply by integrating (4.3) from $z = -1$ to $z = +1$.) We recall that the paths of integration in (4.8) and (4.10) must pass under the branch point at $z = z_c$ (see I for discussion), which implies (cf. (1.7))

$$(z-z_c)^{\nu} = (z_c - z)^{\nu} e^{-i\pi\nu}. \tag{4.11}$$

We may solve the integral equation (4.8) either by developing both R and χ as joint expansions in $z_c - z_0$ and $\nu - \nu_0$ and then equating the coefficients of like powers or by iteration, starting with the zeroth approximation

$$\chi = 1 + O(z_c - z_0, \nu - \nu_0). \tag{4.12}$$

We shall consider in this section only the first approximation, for which purpose we may approximate (4.6) in the form

$$R(z; z_c, \alpha, \nu) = (z_c - z_0) R_a(z; z_0, \alpha, \nu_0) + (\nu - \nu_0) R_b(z; z_0, \alpha, \nu_0), \tag{4.13}$$

where R_a and R_b are both real for $-1 \leqslant z \leqslant 1$. Substituting (4.12) and (4.13) into (4.10), we may place the result in the form

$$X(z_c, \alpha, \nu) = a(z_0, \alpha, \nu_0)(z_c - z_0) + b(z_0, \alpha, \nu_0)(\nu - \nu_0)$$
$$+ O[(z_c - z_0)^2, (z_c - z_0)(\nu - \nu_0), (\nu - \nu_0)^2], \tag{4.14}$$

where

$$a, b = \int_{-1}^{1} (z - z_0)^{\nu_0} g^2(z; \alpha) R_{a,b}(z; z_0, \nu_0, \alpha)\, dz, \tag{4.15}$$

† We have suppressed the explicit appearance of the parameters z_c, α and ν in g, R and χ.

with the path of integration indented under $z = z_0$. We observe that

$$-1 < z_0 < 1, \quad \nu_0 < 1$$

in consequence of Theorems (iii) and (iv). It follows that both a and b are complex numbers except at $\nu_0(\alpha) = 0$.

We have chosen to work in terms of $\nu - \nu_0(\alpha)$, rather than $J - J_0(\alpha)$, in order to avoid the analytical difficulties associated with the branch point of ν *qua* function of J_c. In the final analysis, however, we require a relation between $z_c - z_0(\alpha)$ and $J - J(\alpha)$. Expanding (2.13) about $z_c = z_0$ and $J = J_0$, substituting the linear approximation into (4.14), and requiring the resulting approximation to X to vanish, we obtain

$$z_c - z_0 = \left[J_0\left(\frac{S_0'}{S_0} - \frac{B_0'}{B_0}\right) + \frac{1}{2}\left(\frac{\nu_0 S_0}{B_0}\right)\frac{a(z_0, \alpha, \nu_0)}{b(z_0, \alpha, \nu_0)}\right]^{-1}(J - J_0) \quad (J \to J_0(\alpha)), \tag{4.16}$$

where the subscript zero implies evaluation at $z = z_0$. We conclude from (4.16) that, in so far as a/b is a complex number, the imaginary part of z_c changes sign with $J - J_0(\alpha)$ and hence that the SNM characterized by $J = J_0(\alpha)$ and $z_c = z_0(\alpha)$ marks a transition from a positively damped (stable) to a negatively damped (unstable) mode as the neutral curve $J = J_0(\alpha)$ is crossed.† We emphasize, however, that $J > J_0(\alpha)$ does not necessarily imply stability, although such an *a priori* conclusion is valid by virtue of Theorem (v) if the neutral curve $J = J_0(\alpha)$ is unique and single valued. We also emphasize that the imaginary part of $\nu_0 a/b$ tends to zero as ν_0 tends to any of 0 or ± 1, in consequence of which (4.16) does not provide a uniformly valid approximation to the imaginary part of z_c in the neighbourhoods of these points on the neutral curve. We shall see, in the specific example of the following section, how this difficulty can be circumvented by constructing uniformly valid approximations to R and χ that reduce to those of (4.12) and (4.13) except near $\nu_0 = 0$ or ± 1.

Now let us suppose that both $B(z)$ and $S(z)$ are even functions of z. It then appears likely from symmetry and physical considerations that an SNM will exist for $z_0 = 0$. Assuming $z_0 = 0$, we may infer from (3.6) and (3.7) that q and ϕ_0 are odd and even functions of z, respectively, and hence, from (4.6) and (4.13), that R_a and R_b are even and odd functions. Invoking these properties in (4.15), choosing g according to (3.8), and substituting the resulting expressions for a and b into (4.16), we obtain

$$z_c = 2i\left(\frac{B_0}{\nu_0 S_0}\right)\tan\left(\tfrac{1}{2}\pi\nu_0\right)\left\{\frac{\int_0^1 z^{\nu_0} S^{2\mu} R_b\, dz}{\int_0^1 z^{\nu_0} S^{2\mu} R_a\, dz}\right\}(J - J_0) \quad (J \to J_0(\alpha)), \tag{4.17}$$

which implies the principle of exchange of stabilities (cf. Theorem (vi)).

We remark that, in so far as the unstable eigenvalue that descends to $z_0 = 0$ according to (4.17) is unique, it must remain imaginary, for complex wave speeds

† It should be borne in mind that, in consequence of (4.11), the existence of z_c as an eigenvalue does not imply the admission of its complex conjugate as an eigenvalue.

necessarily occur in pairs.† We shall carry through the argument for the specific example of the following section, where the uniqueness of the SNM can be established.

5. Hølmboe's configuration

If we assume a configuration of infinite extent ($y_1 = -\infty$, $y_2 = \infty$), physical considerations appear to demand $U_1' = U_2' = 0$. The simplest admissible form for the shear then is given by

$$S(z) = 1 - z^2, \tag{5.1}$$

which implies the velocity profile of (1.9). The simplest, corresponding choice for $B(z)$ is

$$B(z) = 1, \tag{5.2}$$

which, in conjunction with (5.1) and (2.3)–(2.5), implies the density profile of (1.10) with $r = 0$. In short,

$$U(y)/V = \lambda(y)/\sigma = \tanh(y/h). \tag{5.3}$$

The corresponding, local Richardson number, as given by (1.5), is

$$J_l(y) = J\cosh^2(y/h), \tag{5.4}$$

which increases monotonically with y^2.

The configuration described by (5.3) was proposed originally by Hølmboe (1960) as mathematically simpler and physically more acceptable than that considered by Drazin (1958).‡ Hølmboe obtained a solution for the stream function by inspection and then determined the neutral curve of (5.12) below. The following derivation proves that this neutral curve is unique and that it comprises all possible SNM's. We shall show subsequently (§ 6 below), however, that this uniqueness is a consequence of the special simplicity of the density profile and not (as might have appeared to be a plausible conjecture) merely of the restrictions that $U'(y)$ and $\lambda'(y)$ be even, positive-definite functions of y in (y_1, y_2) and that $J_l(y)$ increase monotonically with y^2.

Substituting (5.1) and (5.2) into (2.10)–(2.13), we obtain

$$\psi = P\begin{Bmatrix} -1 & z_c & 1 & \infty & \\ \tfrac{1}{2}\alpha & \tfrac{1}{2}(1+\nu) & \tfrac{1}{2}\alpha & \tfrac{1}{2}(1+\tau) & z \\ -\tfrac{1}{2}\alpha & \tfrac{1}{2}(1-\nu) & -\tfrac{1}{2}\alpha & \tfrac{1}{2}(1-\tau) & \end{Bmatrix}, \tag{5.5}$$

$$\nu = (1-4J_c)^{\frac{1}{2}}, \quad J_c = J/(1-z_c^2), \tag{5.6}$$

and

$$\tau = 3. \tag{5.7}$$

Assuming ψ to have the form (3.1), with g defined by (3.8), we then may reduce the differential equation (3.3) to

$$(z-z_c)(1-z^2)\phi'' + [(1+\nu)(1-z^2) - 2(1+\alpha)(z-z_c)z]\phi' + q\phi = 0 \tag{5.8}$$

with

$$q = (\tfrac{5}{2}+\tfrac{1}{2}\nu+\alpha)(\tfrac{1}{2}-\tfrac{1}{2}\nu-\alpha)(z-z_c) + [2\alpha+(3+\nu)(\tfrac{1}{2}-\tfrac{1}{2}\nu-\alpha)]z_c. \tag{5.9}$$

† Complex wave speeds would occur in quartets if the complex conjugate of (1.7) were admitted as a singular neutral mode.

‡ Drazin (1958) considered $U(y) = V\tanh(y/h)$ and $\lambda(y) = \sigma(y/h)$. This implies $B(z) = 1/(1-z^2)$, which renders the exponents with respect to $z = \pm 1$ dependent on both α and J.

The Riemann function with four regular singularities, as described by either (5.5) or (5.8), has been studied in some detail by Heun (1889). We may infer from his results that the solution to (5.8) that is regular at $z = z_c$ can also be regular at $z = \pm 1$ for $\alpha > 0$ if and only if it is a polynomial.

Let us suppose that ϕ is a polynomial of degree n; then, in consideration of the behaviour of ψ at $z = \infty$, our supposition requires

$$\tfrac{1}{2}(1+\nu)+\alpha+n+\tfrac{1}{2}(1\pm\tau) = 0. \tag{5.10}$$

Invoking the restrictions $\alpha > 0$ and $\nu > -1$, we see that (5.10) can be satisfied only if $n = 0$ and the choice $\frac{1}{2}(1-\tau) = -1$ is made for the exponent with respect to $z = \infty$; solving for ν then yields

$$\nu = \nu_0(\alpha) = 1-2\alpha. \tag{5.11a}$$

Substituting (5.11 a) into (5.9) yields $q = 2\alpha z_c$, which must vanish if ϕ = constant is to satisfy (5.8). We conclude that (5.11 a) and

$$z_c = 0 \tag{5.11b}$$

are necessary and sufficient conditions for the existence of a solution to (5.8) that is regular at each of $z = z_c$, $-1 < z_c < 1$, and $z = \pm 1$. We may normalize this solution to

$$\phi \equiv 1. \tag{5.11c}$$

Substituting (5.11 a, b) into (5.6), we obtain

$$J = J_0(\alpha) = \alpha(1-\alpha) \quad (0 < \alpha < 1), \tag{5.12}$$

which is a symmetric parabola in an (α, J)-plane with its maximum at $\alpha = \frac{1}{2}$ and $J = \frac{1}{4}$ (note that $\nu_0 < 0$ for $\frac{1}{2} < \alpha \leqslant 1$). Solutions corresponding to (α, J)-points above this curve must represent stable modes in consequence of Theorems (iii), (v) and (viii) in § 1 above; solutions corresponding to points below this curve must represent unstable modes in consequence of Theorems (iii), (vi) and (viii). Drazin & Howard (1961) have calculated αz_c for these unstable modes through a second-order expansion in α and J, but it appears to be worth while to determine an approximation that is uniformly valid everywhere in the neighbourhood of the neutral curve—in particular, in the (α, J) neighbourhoods of $(0, 0)$, $(\frac{1}{2}, \frac{1}{4})$, and $(1, 0)$, where ν_0 tends to 1, 0, and -1, respectively, and the approximations of (4.12) and (4.13) are inadequate.

We shall proceed as in §§ 3 and 4, but with such modifications as are required to achieve the aforementioned goal of uniform validity. Substituting (5.11 a, b, c) and the associated result $q_0 \equiv 0$ into (4.6) yields

$$R(z, z_c) \equiv q(z, z_c). \tag{5.13}$$

At this point, we modify the formulation of § 4 slightly by developing R and χ in powers of

$$\epsilon = \tfrac{1}{2}(\nu_0-\nu) = \tfrac{1}{2}-\tfrac{1}{2}\nu-\alpha \tag{5.14}$$

and αz_c, rather than $\nu-\nu_0$ and z_c. Substituting (5.11)–(5.14) into (4.8) and setting $\chi = 1$ in the resulting integral, we find that an adequate approximation for our purpose is given by

$$\chi(z) = 1+\epsilon \log(z-z_c)+O(\epsilon\nu, \epsilon z_c, \alpha z_c). \tag{5.15}$$

Substituting (5.11)–(5.15) into (4.10) then yields

$$X(z_c) = \int_{-1}^{1} (1-z^2)^\alpha (z-z_c)^\nu [(3\epsilon-\epsilon^2)(z-z_c)+2(\alpha+2\epsilon)z_c + 3\epsilon^2(z-z_c)\log(z-z_c)]\,dz+\delta, \tag{5.16}$$

where, here and subsequently, δ stands for any error term having the composition

$$\delta = O(\epsilon^2\nu, \epsilon^2 z_c, \epsilon\alpha z_c, \alpha^2 z_c^2). \tag{5.17}$$

We note that $\delta = O(\epsilon^2, \epsilon z_c, z_c^2)$, except in the (α, ν)-neighbourhoods of $(\frac{1}{2}, 0)$, where the retention of terms of $O(\epsilon^2)$ in (5.16) is required, and $(0, 1)$, where the retention of terms in z_c^2 is required (the terms of first order in z_c being $O(\alpha^2)$ as $\alpha \to 0$).

It is readily seen that the integral in (5.16) is an analytic function of z_c in $|z_c| < 1$. Expanding the integrand in z_c, interpreting the phase of z^ν in accordance with (4.11), expressing the resulting integrals in terms of beta functions, and neglecting further terms of the same order as those already comprised by δ, we may reduce the integral as follows:

$$\begin{aligned} X &= \int_{-1}^{1} \{(1-z^2)^\alpha z^\nu[(3\epsilon-\epsilon^2)z+2(\alpha-\epsilon)z_c-(2\alpha+\epsilon)z_c^2 z^{-1}] \\ &\qquad + 3\epsilon^2 z(1-z^2)^{\frac{1}{2}}\log z\}\,dz+\delta, & (5.18a)\\ &= \tfrac{1}{2}(3\epsilon-\epsilon^2)(1-e^{-i\pi\nu})B(\tfrac{1}{2}\nu+1,\alpha+1)+i\pi\epsilon^2+(\alpha-\epsilon)(1+e^{-i\pi\nu})B(\tfrac{1}{2}\nu+\tfrac{1}{2},\alpha+1)z_c \\ &\qquad -(\alpha+\tfrac{1}{2}\epsilon)(1-e^{-i\pi\nu})B(\tfrac{1}{2}\nu,\alpha+1)z_c^2+\delta, & (5.18b)\\ &= \tfrac{3}{2}\epsilon(1+e^{2i\pi\alpha})B(\tfrac{3}{2}-\alpha,1+\alpha)-i\pi\epsilon^2+\alpha(1-e^{2i\pi\alpha})B(1-\alpha,1+\alpha)z_c \\ &\qquad -2(2\alpha+\epsilon)z_c^2+\delta. & (5.18c)\end{aligned}$$

Substituting ν from (5.6) into (5.18c), we find that the dependence of ν on z_c is significant (within the approximations already invoked) only in the neighbourhood of $\alpha = 0$, where $J = O(\alpha)$ as $\epsilon \to 0$ and

$$\epsilon = J(1-z_c^2)^{-1}-\alpha+O[\alpha^2(1-z_c^2)^{-2}]. \tag{5.19}$$

Taking this into account, we find that an approximation to X that is uniformly valid with respect to α and J is given by

$$X(z_c) = \tfrac{3}{2}E(1+e^{2i\pi\alpha})B(\tfrac{3}{2}-\alpha,1+\alpha)-i\pi E^2 + \alpha(1-e^{2i\pi\alpha})B(1-\alpha,1+\alpha)z_c-2\alpha z_c^2+\delta, \tag{5.20}$$

where

$$E = \tfrac{1}{2}-\alpha-(\tfrac{1}{4}-J)^{\frac{1}{2}} \quad (0 \leqslant \alpha < \tfrac{1}{2}), \tag{5.21a}$$

$$= \tfrac{1}{2}-\alpha+(\tfrac{1}{4}-J)^{\frac{1}{2}} \quad (\tfrac{1}{2} < \alpha \leqslant 1). \tag{5.21b}$$

We now write

$$\alpha z_c = i\gamma, \tag{5.22}$$

where the real part of γ is the (dimensionless) exponential rate of growth of the disturbance. Multiplying (5.20) through by $\frac{1}{3}\alpha\exp(-i\pi\alpha)/B(\frac{3}{2}-\alpha,1+\alpha)$ and equating the result to zero, we then obtain

$$\gamma^2+\tfrac{1}{2}\alpha B(\tfrac{1}{2},1-\alpha)\sin(\pi\alpha)\gamma+\alpha\cos(\pi\alpha)E-\tfrac{1}{4}\pi E^2+\alpha\delta = 0. \tag{5.23}$$

The quadratic equation (5.23) has real roots in a finite neighbourhood of the neutral curve $J = J_0(\alpha)$; one and only one of these is positive if and only if

$J < J_0(\alpha)$; and this root vanishes uniformly, and changes sign, with $J_0(\alpha)-J$, in accordance with (4.17). Suitable approximations to the unstable roots are given by

$$\gamma = [(1-4J)^{\frac{1}{2}}+2\alpha-1]\cot(\pi\alpha)/B(\tfrac{1}{2},1-\alpha) \quad (0<\alpha<\tfrac{1}{2}) \tag{5.24}$$

and

$$\gamma = [(1-4J)^{\frac{1}{2}}-2\alpha+1]/B(\tfrac{1}{2},\alpha-\tfrac{1}{2}) \quad (\tfrac{1}{2}<\alpha\leqslant 1) \tag{5.25}$$

except in the (α, J) neighbourhoods of $(0,0)$ and $(\frac{1}{2},\frac{1}{4})$. Suitable approximations in these neighbourhoods are given by

$$\gamma = \{\alpha[J_0(\alpha)-J]+(\tfrac{1}{2}\pi\alpha^2)^2\}^{\frac{1}{2}}-\tfrac{1}{2}\pi\alpha^2+O(\alpha^3),$$
$$J < J_0(\alpha)+\tfrac{1}{4}\pi^2\alpha^3+O(\alpha^4), \tag{5.26}$$

and

$$\gamma = J_0(\alpha)-J, \quad \alpha\to\tfrac{1}{2}, \quad J\to\tfrac{1}{4}. \tag{5.27}$$

We note that Drazin & Howard's (1961, equation (43)) approximation reduces to (5.26) as $\alpha\to 0$ but gives values of γ that are low by factors of $2/\pi$ and $\frac{1}{2}$ as $\alpha\to\frac{1}{2}$ and 1, respectively, when compared with (5.27) and (5.25). It therefore appears likely that the wave number for maximum instability, $\alpha_M(J)$, must lie to the right of the curve plotted in their figure 3.†

We conclude this section by remarking that the unstable eigenvalue determined from (5.20) must remain imaginary, and hence that the corresponding unstable mode must remain stationary ($c_r\equiv 0$) for $0\leqslant J\leqslant J_0(\alpha)$, rather than merely in the neighbourhood of $J=J_0(\alpha)-$. This follows from the considerations that: (*a*) the singular neutral eigenvalue $z_c=0$ is unique; (*b*) z_c must be a continuous function of α and J within the semicircle of Theorem (ii) and must tend to $z_c=0$ as $J\to J_0(\alpha)-$; and (*c*) complex wave speeds necessarily occur in pairs (if c_r+ic_i is an eigenvalue, so also must be $-c_r+ic_i$ by virtue of symmetry).

We emphasize that statement (*b*) does not hold for damped waves. For example, (5.23) yields a stable, as well as an unstable, eigenvalue for $J<J_0(\alpha)$, and our analysis does not preclude the existence of more than one such (stable) eigenvalue; moreover, (5.23) yields a pair of stable eigenvalues if $J>J_0(\alpha)$, which tend to $z_c=0$ and $-i\pi\alpha$ as $J\to J_0(\alpha)+$ and are complex if

$$J_0(\alpha)+\tfrac{1}{4}\pi^2\alpha^3 < J \ll 1.$$

6. A configuration with multivalued neutral curves

A simple generalization of Hølmboe's configuration that leaves the differential equation for ψ basically unchanged is given by

$$B(z) = 1-r+3rz^2, \quad -\tfrac{1}{2}\leqslant r<1, \tag{6.1}$$

which, in conjunction with (5.1) and (2.3)–(2.5), implies the density profile of (1.10).‡ The corresponding, local Richardson number, as given by (1.5), is

$$J_l(y) = J[(1-r)\cosh^2(y/h)+3r\sinh^2(y/h)] \geqslant (1-r)J. \tag{6.2}$$

† Howard (private communication) concurs in this opinion and informs us that Dr A. Michalke in Berlin has obtained $\alpha_M(0)=0{\cdot}4436$ from a direct, numerical solution of the eigenvalue problem posed by a homogeneous shear flow with the velocity profile of (1.9) above.

‡ A closely related configuration has been considered by Drazin & Howard (private communication).

The restriction $-\frac{1}{2} \leqslant r < 1$ guarantees that $B(z) > 0$ for $-1 < z < 1$ and implies that $J_l(y)$ increases monotonically with y^2.

The configuration described by (1.9) and

$$\lambda(y) = \sigma \tanh^3 (y/h), \tag{6.3}$$

corresponding to $r = 1$ in (1.10), has been considered by Garcia (1961). He assumed $c = 0$ and obtained results equivalent to (6.9) and (6.10*a*) below with $r = 1$ therein. We remark that, in this very special case, the singularity of the differential equation (2.1) at $y = y_c = 0$ is only apparent. See also the remarks at the end of the paragraph following (6.14) below.

Substituting (5.1) and (6.1) into (2.10)–(2.13), (3.8) and (3.9), we obtain the Riemann function (5.5), with

$$\nu = (1-4J_c)^{\frac{1}{2}}, \quad J_c = J(1-r+3rz_c^2)/(1-z_c^2) \tag{6.4}$$

and

$$\tau = (9+12rJ)^{\frac{1}{2}}, \tag{6.5}$$

and Heun's differential equation (5.8) with

$$q = [\tfrac{1}{4}(\tau^2-\nu^2)-2](z+z_c)+[1-\nu-(1+\nu)\alpha]z-(\alpha+\alpha^2)(z-z_c). \tag{6.6}$$

It follows, as in § 5, that the solution for an SNM must be a polynomial in z and that the exponents of the differential equation (5.8) for a polynomial solution of degree n must be constrained according to (5.10), which we now rewrite in the form

$$\alpha+n = \alpha_n(J, z_c; r) = \tfrac{1}{2}(\tau-\nu)-1. \tag{6.7}$$

If $r < 0$ the maximum value of τ is 3 and the right-hand side of (6.7) cannot exceed 1 (since $\nu > -1$). It then follows, from the restriction $\alpha > 0$, that only $n = 0$ is admissible, and this implies $z_c = 0$, just as in § 5. If $r > 0$ the right-hand side of (6.7) may exceed 1 (since τ may exceed 3), but we anticipate (and shall demonstrate subsequently) that $z_c = 0$ gives the most critical SNM, and hence the stability boundary, for any even value of n and that no SNM exists for odd values of n if $r < 0{\cdot}947$.

Considering first, then, stationary SNM's, we substitute (6.6) into (5.8), eliminate ν through (6.7), and set $z_c = 0$ to obtain

$$z(1-z^2)\phi''+[(\tau-2\alpha-2n-1)(1-z^2)-2(1+\alpha)z^2]\phi'+n(\tau-n)z\phi = 0. \tag{6.8}$$

Introducing z^2 as the independent variable, we may transform (6.8) to the hypergeometric equation and obtain the polynomial solutions

$$\phi = {}_2F_1(-\tfrac{1}{2}n, \tfrac{1}{2}\tau-\tfrac{1}{2}n; \tfrac{1}{2}\tau-\alpha-n; z^2) \quad (n = 0, 2, \ldots), \tag{6.9}$$

for n even. There are no solutions to (6.8) that are regular at each of $z = z_c = 0$ and $z = \pm 1$ if n is odd.

Substituting (6.4) and (6.5) into (6.7), setting $z_c = 0$, and indicating explicitly that ν may be either positive or negative, we obtain the neutral curves corresponding to the SNM's of (6.9) in the form

$$\alpha_n(J,0;r) = \tfrac{1}{2}(9+12rJ)^{\frac{1}{2}} \mp \tfrac{1}{2}[1-4(1-r)J]^{\frac{1}{2}}-1, \tag{6.10a}$$

$$= \frac{3}{2}\left[1+\frac{4}{3}\left(\frac{r}{1-r}\right)J_c\right]^{\frac{1}{2}} \mp \tfrac{1}{2}(1-4J_c)^{\frac{1}{2}}-1, \tag{6.10b}$$

$$= \alpha_n(J,0;0)+\left(\frac{r}{1-r}\right)J_c-\frac{1}{3}\left(\frac{r}{1-r}\right)^2 J_c^2+\ldots, \tag{6.10c}$$

where

$$J_c = (1-r)J \quad (z_c = 0). \tag{6.11}$$

The two branches of α_n given by (6.10) have the limiting values 0 and 1 at $J = 0$ and join at $J_c = \frac{1}{4}$. There are no solutions for $J_c > \frac{1}{4}$, as we also may infer directly from Theorem (v) and the fact that $J_l(y) \geqslant J_c$. The maximum value of α_n is simply 1 if $r \leqslant \frac{1}{2}$, but if $r > \frac{1}{2}$

$$(\alpha_n)_{\max} = \left[\frac{(3-2r)(1+2r)}{4r(1-r)}\right]^{\frac{1}{2}} - 1 \quad (\tfrac{1}{2} \leqslant r < 1) \tag{6.12a}$$

at

$$J_c = 3(2r-1)/4r(1+2r). \tag{6.12b}$$

Invoking the requirement $\alpha_n > n$ ($\alpha > 0$), we then may determine the maximum value of n, say N, for which the SNM of (6.9) is admissible from the inequality

$$N < \left[\frac{(3-2r)(1+2r)}{4r(1-r)}\right]^{\frac{1}{2}} - 1 < N+1, \tag{6.13}$$

or, equivalently,

$$\left[\frac{(N-1)(N+3)}{N(N+2)}\right]^{\frac{1}{2}} < 2r-1 < \left[\frac{N(N+4)}{(N+1)(N+3)}\right]^{\frac{1}{2}}. \tag{6.14}$$

Summing up, we infer from the foregoing results that the neutral curve $J = J_0(\alpha)$ is no longer single valued if $r > \frac{1}{2}$ and that there are at least $\frac{1}{2}N+1$ distinct neutral curves (each having two branches), say $J = J_0(\alpha;\, n)$, if r satisfies (6.14) with N even; in particular, $N = 0$ if $r \leqslant 0{\cdot}895$, $N = 2$ if $0{\cdot}895 < r < 0{\cdot}968$, and $N \to \infty$ as $r \to 1$. Increasing J then may have a destabilizing effect for some range of α, although it remains true that $J_c > \frac{1}{4}$ is a sufficient condition for stability for any discrete value of α and a necessary condition for stability for a complete spectrum of α ($\alpha \geqslant 0$).

The neutral curves for $r = -\frac{1}{4}$, 0 and $\frac{3}{4}$ are compared in an (α, J_c) plane in figure 1; those for $r = 0{\cdot}9$ are plotted in an (α, J_c) plane in figure 2; and those for $r = 0{\cdot}9$ and 1 are compared in an (α, J) plane in figure 3. We emphasize that the neutral curve $J = J_0(\alpha;\, n)$ is a stability boundary only for the nth mode; e.g. for $r = 0{\cdot}9$ (see figure 2) the $n = 0$ mode is stable for $\alpha = 0$ and $J > 0$, but the $n = 2$ mode is unstable for $\alpha = 0$ and $0{\cdot}20 < J_c < 0{\cdot}25$.

Stability is impossible for $r = 1$, since unstable modes exist for every (α, J)-point in $\alpha > 0$ and $J > 0$. This limiting case violates our initial restriction to positive-definite density gradients, however, and instability is a consequence of the fact that $J_c = 0$.

It is of interest to compare the result (6.10a) with that given by Drazin & Howard's (1961) small-α approximation. Substituting (1.9) and (1.10) into their (27), we obtain

$$J = \alpha - (1 - 2r + \tfrac{2}{3}r^2)\,\alpha^2 + O(\alpha^3), \tag{6.15}$$

which agrees with (6.10) within the indicated approximation. This approximation happens to be exact for $r = 0$, as observed by Drazin & Howard, but if $r > \frac{1}{2}$ it is obviously quite wide of the mark for non-small α and fails to reproduce the bulge characterized by (6.12).

We now proceed to consider the existence of nonstationary SNM's for $r > 0$. We shall find it instructive, for this purpose, to examine (6.7) in an (α_n, τ)-plane.

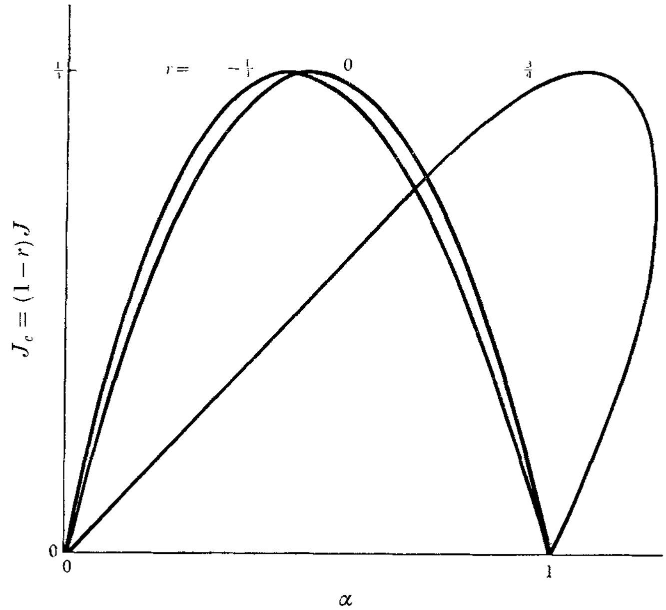

FIGURE 1. The neutral curves given by (6.10*b*) for $n = 0$ and $r = -\frac{1}{4}$, 0 and $\frac{3}{4}$.

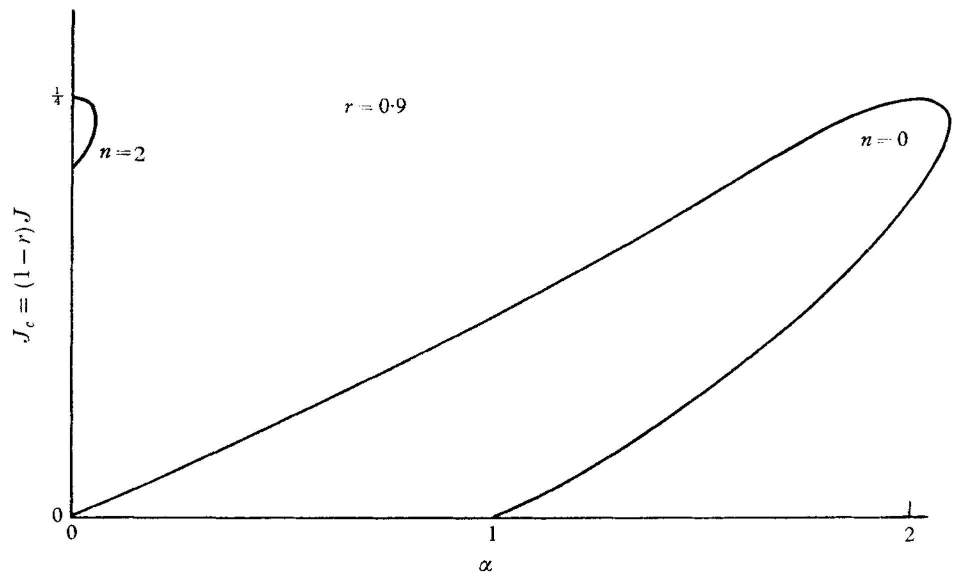

FIGURE 2. The neutral curves given by (6.10*b*) for $n = 0$ and 2 and $r = 0{\cdot}9$.

Eliminating J between (6.4) and (6.5), we may express ν in terms of τ and a single parameter s:

$$\nu = [1 - s(\tau^2 - 9)]^{\frac{1}{2}}, \tag{6.16}$$

$$s = (s_0 + z_c^2)/(1 - z_c^2), \quad s_0 = (1-r)/3r. \tag{6.17}$$

Substituting (6.16) into (6.7), we may transform the result to

$$4(\alpha_n + 1)^2 - 4(\alpha_n + 1)\tau + (1+s)\tau^2 = 1 + 9s, \tag{6.18}$$

which represents a family of ellipses with parameter s ($r > 0$ implies $s \geqslant s_0 > 0$). Each member of this family passes through $\alpha_n = 0$ and $\alpha_n = 1$ at $\tau = 3$ ($J = 0$) and lies between the straight lines $\tau = 2(\alpha_n + 1) \pm 1$ for

$$3 < \tau < (9+s^{-1})^{\frac{1}{2}} \quad (0 < J_c < \tfrac{1}{4});$$

moreover, the two branches of α_n, *qua* function of τ, must join on the straight line $\tau = 2(\alpha_n + 1)$. The maximum value of α_n for $J > 0$ is given by

$$(\alpha_n)_{\max} = 1 \quad (s \geqslant \tfrac{1}{3}), \tag{6.19a}$$

$$= \tfrac{1}{2}[(1+s)(1+9s)/s]^{\frac{1}{2}} - 1 \quad (0 < s \leqslant \tfrac{1}{3}). \tag{6.19b}$$

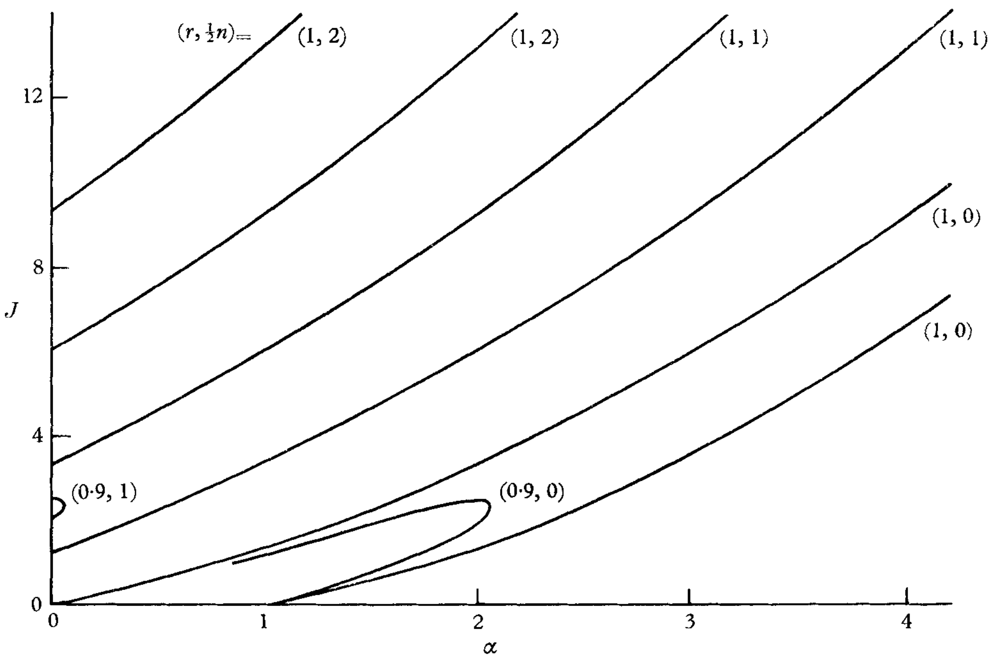

FIGURE 3. The neutral curves given by (6.10a) for $r = 0{\cdot}9$ and 1.

Invoking the requirements $\alpha_n > n$ ($\alpha > 0$) and $J > 0$, we infer from (6.19a) that $n = 0$ is the only admissible possibility for $s \geqslant \frac{1}{3}$. This implies $z_c = 0$, as in §5, whence $s \equiv s_0 \geqslant \frac{1}{3}$ implies $0 \leqslant r \leqslant \frac{1}{2}$. We also may infer, from the considerations that the right-hand side of (6.19b) is a monotonically decreasing function of s and that s is a monotonically increasing function of z_c^2, that $z_c = 0$ yields an upper bound to n, namely N, as given by (6.13) and (6.14). This does not guarantee the existence of an SNM for $n = N$, however, since $z_c = 0$ is not generally admissible as an eigenvalue unless n is even (see below).

Now let us suppose that

$$0 < z_c^2 < 1, \quad s_0 < s < \tfrac{1}{3}. \tag{6.20}$$

The neutral curve (on which z_c is not, in general, constant) then must lie inside the ellipse obtained by setting $s = s_c$ ($z_c = 0$) in (6.18). It follows that the most critical SNM must correspond to $z_c = 0$, if this eigenvalue is admissible, and hence that $z_c = 0$ gives the stability boundary for even values of $n \leqslant N$, albeit nonstationary SNM's may exist for such values of n.

To proceed further, we must determine those values of z_c that, in conjunction with the constraint (6.7), permit polynomial solutions of (5.8), say

$$\phi = \sum_{m=0}^{n} c_m z^m. \tag{6.21}$$

Substituting (6.6) into (5.8) and eliminating ν from the result with the aid of (6.7), we obtain the differential equation to be satisfied by (6.21) in the form

$$(z-z_c)(1-z^2)\phi'' + [(\tau-2\alpha-2n-1)(1-z^2)-2(1+\alpha)z(z-z_c)]\phi' + \{n(\tau-n)(z-z_c)+[\tau\alpha+(2n+1)(\tau-\alpha)-(2n^2+2n+3)]z_c\}\phi = 0. \tag{6.22}$$

Substituting (6.21) into (6.22) and equating the coefficients of z^m separately to zero for $m = 0, 1, \ldots, n$ (the coefficient of z^{n+1} vanishes identically by virtue of (6.7)) then yields $n+1$ linear equations in $c_0, c_1, \ldots, c_n$. Observing that each of these equations is linear in z_c, we infer from the requirement that their determinant vanish that there exist $n+1$ distinct polynomial solutions to (6.22), corresponding to $n+1$ distinct values of z_c.† We also may infer, from (3.6) and (3.7), that these eigenvalues would comprise $\frac{1}{2}(n+1)$ pairs of equal and opposite values of z_c if n is odd or $\frac{1}{2}n$ such pairs plus $z_c = 0$ if n is even. It is by no means clear, however, that each of the non-zero values of z_c^2 determined in this fashion is both in the singular-neutral interval (0, 1) and compatible with the satisfaction of (6.7) for $\alpha > 0$ and $J > 0$.

The explicit determination of any of the nonstationary SNM's and its associated neutral curve in an (α, J)-plane appears to lead to an intractable algebraic problem. We shall rest content with a demonstration that at least one such SNM exists, remarking that this result suffices to disprove the not uncommon conjecture that sufficient conditions for the non-existence of nonstationary SNM's, and hence for the validity of the principle of exchange of stabilities, are that $U'(y)$ and $\lambda'(y)$ be positive-definite, even functions of y in $(-\infty, \infty)$.‡

Returning to (6.21), substituting it into (6.22), setting $n = 1$, and requiring the determinant of the resulting equations in c_0 and c_1 to vanish, we obtain

$$z_c^2 = \frac{(\tau-2\alpha-3)(\tau-1)}{(\tau-3)(\alpha+2)[(\tau-3)(\alpha+2)+2(\alpha+1)]}. \tag{6.23}$$

† We note that this last result is in agreement with Heine's (1878, vol. 1, pp. 472 ff) theorem for Fuchsian differential equations; however, the direct application of Heine's theorem would require the coefficients of ϕ'' and ϕ' in (5.8), and hence z_c, to be assigned *a priori* and then would imply the existence of $n+1$ determinations of q. The corresponding ϕ are *Heun polynomials* (Erdelyi, Magnus, Oberhettinger & Tricomi 1953, vol. 1, p. 218 and vol. 3, pp. 60 ff).

‡ L. N. Howard (private communication) has demonstrated that the conjecture is definitely false for the configuration obtained by replacing $U(y) = y$ by $U(y) = 0$ for $|y| < h$ in (1.11). The resulting configuration has a nonstationary SNM, and the principle of exchange of stabilities does not apply; an SNM also exists, but it is non-singular. This is an extreme example, however, since $U'(y)$ and $\lambda'(y)$ are neither positive-definite nor bounded, and the concentrated vortex sheets at $y = \pm 1$ are susceptible to Kelvin–Helmholtz instability. Hølmboe (1962) has obtained similar results for the configuration obtained by replacing $\lambda(y)$ in (1.11) by $\lambda(y) = \sigma \operatorname{sgn} y$.

Eliminating s between (6.17) and (6.18), we also must have

$$z_c^2 = \frac{(\tau-2\alpha-3)(2\alpha+5-\tau)-s_0(\tau^2-9)}{(\tau-2\alpha-3)(2\alpha+5-\tau)+(\tau^2-9)}. \tag{6.24}$$

A neutral curve exists if (6.23) and (6.24) can be satisfied simultaneously for $\alpha > 0$, $\tau > 3$ and $0 < z_c^2 < 1$. We know from the preceding discussion that such a curve must lie inside the ellipse obtained by setting $s = s_0$ ($z_c = 0$) in (6.18) and that its two branches (for α *qua* function of τ) must join at least once on the straight line $\tau = 2\alpha+4$ (it might be either a closed loop, intersecting $\tau = 2\alpha+4$ twice to give maximum and minimum values of τ, or an open loop terminated by $\alpha = 0$). We also know, from Theorems (vii) and (viii), that the neutral curve must be continuous in $\alpha > 0$, and we therefore may infer its existence or non-existence from the existence or non-existence of at least one simultaneous solution of (6.23) and (6.24) for $\tau = 2\alpha+4$ and $\alpha > 0$.

Substituting $\tau = 2\alpha+4$ in (6.23) and (6.24), we obtain

$$z_c^2 = \frac{(2\alpha+3)}{(2\alpha+1)(\alpha+2)(2\alpha^2+7\alpha+4)} = \frac{1-s_0(2\alpha+1)(2\alpha+7)}{1+(2\alpha+1)(2\alpha+7)} \quad (\tau = 2\alpha+4). \tag{6.25a,b}$$

Eliminating z_c^2 between (6.25a,b), we may place the resulting sextic equation for α in the alternative forms

$$16s_0\alpha^6+160s_0\alpha^5+4(150s_0-1)\alpha^4+16(66s_0-1)\alpha^3+(905s_0-3)\alpha^2 \\ +2(183s_0+15)\alpha+8(7s_0+2) = 0, \tag{6.26}$$

or

$$s_0 = \frac{4\alpha^4+16\alpha^3+3\alpha^2-30\alpha-16}{(\alpha+2)(2\alpha+1)^2(2\alpha+7)(2\alpha^2+7\alpha+4)}. \tag{6.27}$$

Invoking Descartes' rule of signs, we find that (6.26) has no positive real roots if $s_0 > 1/66$ ($r < 22/23 = 0{\cdot}9565$) and either 0 or 2 such roots if $s_0 < 1/66$. We also find that the right-hand side of (6.27) has one and only one positive real zero, $\alpha \doteqdot 1{\cdot}33$, and that $s_0 > 0$ for $\alpha > 1{\cdot}33$. It follows that there exists a positive value of s_0, say s_{0*} ($r = r_*$), such that there exist exactly two positive real values of α for which (6.25a,b) can be satisfied if $0 < s_0 < s_{0*} \leqslant 1/66$ ($1 > r > r_* > 0{\cdot}956$). The corresponding neutral curve would be a closed loop having a minimum value of τ in excess of $\tau = 2(2\times 1{\cdot}33+4) = 13{\cdot}33$, corresponding to a minimum value of J in excess of $14{\cdot}1$. The numerical calculation of r_* would be straightforward, but it suffices for our purpose to have demonstrated its existence and established the lower bound $0{\cdot}956$.

Now let us return to (6.14), which yields the upper n-bounds $N = 0, 1, 2, 3, \ldots$ for r in $[0, \frac{1}{2}]$, $(\frac{1}{2}, 0{\cdot}895]$, $(0{\cdot}895, 0{\cdot}947]$, $(0{\cdot}947, 0{\cdot}968]$, ... But, from the preceding paragraph, no SNM exists for $n = 1$ if $r < 0{\cdot}956$. We conclude that the admissible values for n are 0 for $-\frac{1}{2} \leqslant r \leqslant 0{\cdot}895$ and 0 and 2 for $0{\cdot}895 < r \leqslant 0{\cdot}947$, that the most critical SNM's for these values of n are stationary ($z_c = 0$), that the corresponding stability boundaries are given by (6.10), and that the principle of exchange of stabilities holds relative to these boundaries (see penultimate paragraph of § 4).

7. Conclusions

We conclude that:

(ix) The existence of an SNM implies the existence of a contiguous unstable mode in an (α, J)-plane.

(x) The neutral curve $J = J_0(\alpha)$ is not generally single-valued; moreover, several distinct neutral curves may exist for a given configuration, each such curve corresponding to a distinct SNM.

(xi) The principle of exchange of stabilities holds for a stationary SNM if $U'(y)$ and $\lambda'(y)$ are positive-definite, symmetric functions of y that possess analytic continuations into a complex-U plane, but nonstationary SNM's may exist even though these restrictions are satisfied.

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