# On the stability of heterogeneous shear flows

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Small perturbations of a parallel shear flow $U(y)$ in an inviscid, incompressible fluid of variable density $\rho_0(y)$ are considered. It is deduced that dynamic instability of statically stable flows ($\rho_0'(y) < 0$) cannot be other than exponential, in consequence of which it suffices to consider spatially periodic, travelling waves. The general solution of the resulting differential equation is considered in some detail, with special emphasis on the Reynolds stress that transfers energy from the mean flow to the travelling wave. It is proved (as originally conjectured by G. I. Taylor) that sufficient conditions for stability are $U'(y) \neq 0$ and $J(y) > \frac{1}{4}$ throughout the flow, where $J(y) = -g\rho_0'(y)/\rho_0(y)U'^2(y)$ is the local Richardson number. It also is pointed out that the kinetic energy of a normal mode in an ideal fluid may be infinite if $0 < J(y_c) < \frac{1}{4}$, where $U(y_c)$ is the wave speed.

## 1. Introduction

We shall consider here the stability of a heterogeneous (or stratified) shear flow—namely a parallel shear flow $U(y)$ in an inviscid, non-heat-conducting, incompressible fluid of density $\rho_0(y)$. The principal measure of stability, in so far as the buoyancy effects of the density gradient override its inertial effects, is the Richardson number

$$J(y) = \beta g (dU/dy)^{-2}, \tag{1.1}$$

where

$$\beta(y) = -d/dy \log \rho_0(y) = -\rho_0'(y)/\rho_0(y) \tag{1.2}$$

is a measure of the static stability of the density stratification ($\beta < 0$ implies static instability). The inertial effects of density gradient are measured by $\beta l$, where $l$ is a characteristic length; neglecting these effects, as in meteorological and oceanographic problems, implies $|\beta| l \ll 1$.

Analytical studies of heterogeneous shear flow, although finding antecedents in the work of Kelvin and Rayleigh on hydrodynamic stability, may be dated from G. I. Taylor's Adams Prize essay of 1915, which was published concurrently with a closely related investigation by Goldstein (1931). These two papers, dealing primarily with specific flow configurations (see below), were followed closely by Synge's (1933) study of the general boundary-value problem. This important paper appears to have escaped the notice of many later workers,† who have rediscovered several of Synge's results. Subsequent work has been largely in the hands of meteorologists (often in difficult-to-obtain sources),

† Including the present writer, who is indebted to Dr Drazin for bringing Synge's work to his attention.

with special reference to the problems of atmospheric stability and mountain-lee-wave formation (see Hølmboe (1957) for a survey of the latter problem).

The primary goal of the present investigation is a proof that $U'(y) \neq 0$ and $J(y) > \frac{1}{4}$ are sufficient conditions for the stability of a heterogeneous shear flow (Theorem X in §5), as originally conjectured by Taylor (1915/31). We shall follow the usual, normal-mode approach on the ground that dynamic instability of a statically stable ($\beta > 0$) shear flow of an ideal fluid cannot be other than exponential. This is not to say that the normal-mode approach is completely adequate for an ideal fluid, however, and we shall call attention to the striking fact, apparently overlooked by previous workers, that the energies of those modes that are comprised by the neutral-stability boundary *may* be infinite. We also remark that none of the existing investigations of *continuous* shear and density profiles has proved that unstable modes exist and are contiguous to the loci of neutral modes that are asserted to form the stability boundaries. Such a proof is much to be desired, especially in consequence of the aforementioned possibility of infinite energy for the neutral modes. (The energy of an unstable mode is found to be finite at any fixed time, although tending to infinity exponentially in time.)

Throughout our investigation, we shall place special emphasis on the Reynolds stress that transfers energy from the mean flow to the disturbance, following closely the corresponding treatment of homogeneous shear flow (see Lin 1955: this reference will be denoted subsequently by L, followed by the appropriate section or equation number). This is, in some instances, less direct than the application of standard oscillation theorems to the boundary-value problem (as in Synge's paper), but it does throw additional light on the problem and may point the way to further progress in the investigation of unstable modes.

Before proceeding with the mathematical development, we shall outline the known results for a few, representative configurations, assuming $|\beta| l \ll 1$ except as noted.

Taylor (1931) considered a semi-infinite flow, above a horizontal wall, with $U'$ and $\beta$ constant. He concluded that only neutral waves could exist for $J > \frac{1}{4}$ and that no (harmonic) waves could exist for $0 < J < \frac{1}{4}$. These results were clarified and extended by Eliassen, Høiland & Riis (1953), who considered flow between two parallel walls with $U'$ and $\beta$ constant. They attacked the initial-value problem and showed that a disturbance originating from arbitrary initial conditions would behave asymptotically like $t^{\frac{1}{2}(\nu-1)}$, $\nu = (1-4J)^{\frac{1}{2}}$, for $-\frac{3}{4} < J < \frac{1}{4}$ (and hence be unstable for $J < 0$) but would be exponentially unstable only for $J < -\frac{3}{4}$; for the semi-infinite case, this asymptotic behaviour holds for $-2 < J < \frac{1}{4}$, with exponential instability for $J < -2$. The initial-value problem for the semi-infinite case also has been solved by Case (1960). His analysis is less general than that of Eliassen *et al.* in considering only $J > 0$, but more general in allowing for the inertial effects of density variation.

Goldstein (1931) considered an imbedded shear layer of thickness $2h$, with both $U'$ and $\beta$ constant in $|y| < h$ and vanishing in $|y| > h$. He concluded that harmonic perturbations would be stable for $J > \frac{1}{4}$ and unstable for all wavelengths for $0 < J < \frac{1}{4}$.

Drazin (1958*a*) has considered a more realistic model for an imbedded shear layer, with $U(y) = U_0' h \tanh(y/h)$ and $\beta = \text{const}$. He too concluded that harmonic perturbations would be stable for $J_0 = (\beta g / U_0'^2) < \frac{1}{4}$, but predicted instability only for a finite range of wavelengths for $0 < J_0 < \frac{1}{4}$. Drazin concluded that his agreement with Goldstein on the value of the critical Richardson number, namely $J = \frac{1}{4}$, was coincidental (that it is not coincidental follows from Theorem X below, together with the fact that $J(y) \geqslant J_0$ in Drazin's model). Menkes (1959, 1960) has extended Drazin's analysis by including the inertial effects of density variation, which are found to be stabilizing. Hølmboe (1960) has modified Drazin's model by letting $\beta = \beta_0 \operatorname{sech}^2(y/h)$. This leads to the neutral curve $J_0 = \alpha(1-\alpha)$ in place of Drazin's result $J_0 = \alpha^2(1-\alpha^2)$, where $\alpha = kh$ and $k$ is the wave-number.

Neither Drazin's model nor Hølmboe's modification thereof yields instability for $\alpha > 1$, in striking disagreement with Goldstein's result for $0 < J < \frac{1}{4}$. It seems clear that the discontinuities in $U'$ and $\beta$ at $y = \pm h$ invalidate Goldstein's result for sufficiently small wavelengths (large $\alpha$), but it is not clear why his model should predict instability for very long wavelengths (small $\alpha$). Indeed, we should expect the result for sufficiently long wavelengths to resemble that for Kelvin–Helmholtz instability of a vortex sheet. Assuming a small discontinuity $\Delta\rho_0$ ($\ll \rho_0$) in density and a discontinuity $\Delta U$ in velocity, the latter model predicts stability for (Lamb 1932, §232)

$$k < 2g\Delta\rho_0/\rho_0(\Delta U)^2, \tag{1.3}$$

or

$$\alpha = kh < g\left(\frac{\Delta\rho_0}{2h\rho_0}\right)\left(\frac{2h}{\Delta U}\right)^2 = J. \tag{1.4}$$

Curiously enough, Goldstein did obtain (1.4), as $\alpha \to 0$, for the somewhat cruder model having $U'$ constant in $|y| < h$ and vanishing in $|y| > h$ and density constant except for equal discontinuities at $y = \pm h$ (in contrast to the continuous density profile for the model described in the penultimate paragraph). Drazin's model predicts stability for $\alpha < J_0^{\frac{1}{2}}$ as $\alpha \to 0$, but Hølmboe's modification leads to (1.4). The anomalous character of Drazin's result evidently must be charged to the evanescence of his density profile as $y \to \infty$.†

## 2. Equations of motion

The basic equations of motion for our model are the Euler equation

$$\mathbf{q}_t + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\rho^{-1}\nabla p - g\nabla y, \tag{2.1}$$

the condition of incompressibility

$$\rho_t + (\mathbf{q}\cdot\nabla)\rho = 0, \tag{2.2}$$

and the equation of continuity

$$\nabla\cdot\mathbf{q} = 0, \tag{2.3}$$

where $\mathbf{q}$, $p$ and $\rho$ denote vector velocity, pressure and density. We shall consider

† Drazin (private communication) has pointed out that the limiting forms $\alpha \sim J_0^{\frac{1}{2}}$ and $\alpha \sim J_0$, $\alpha \to 0$, for the neutral stability boundaries in his original model and Holmboe's modification thereof may be deduced from dimensional considerations.

only small, two-dimensional disturbances, noting that three-dimensional disturbances of the same wavelength generally are more stable (Yih 1955). Let

$$\mathbf{q} = (U+u, v), \quad \rho = \rho_0 - \rho_0' \eta, \tag{2.4 a, b}$$

where $u$ and $v$ denote the $x$- and $y$-components of the perturbation velocity and $\eta$ denotes the vertical displacement of a particle from its initial position.† Introducing (2.4$a, b$) in (2.1)–(2.3) and neglecting terms of second order in $u$, $v$ and $\eta$, we obtain the linearized equations

$$\rho_0\left(\frac{Du}{Dt} + U'v\right) = -(p-p_0)_x, \tag{2.5a}$$

$$\rho_0\left(\frac{Dv}{Dt} + \beta g \eta\right) = -(p-p_0)_y, \tag{2.5b}$$

$$\frac{D\eta}{Dt} = v, \tag{2.6}$$

and

$$u_x + v_y = 0, \tag{2.7}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x},$$

$\beta$ is defined by (1.2), subscripts denote partial differentiation, and primes denote differentiation with respect to $y$.

Taking the scalar product of (2.5$a, b$) with $(u, v)$, substituting $v$ from (2.6) in the product $\beta g v \eta$, multiplying (2.7) by $p - p_0$, adding the results, and integrating over the domain under consideration, say $(x_1, x_2)$ and $(y_1, y_2)$, we obtain the energy-transfer equation

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} \rho_0 \left[\frac{1}{2}\frac{D}{Dt}(u^2+v^2+\beta g \eta^2) + U'uv\right] dx\,dy = -\int_{x_1}^{x_2}\int_{y_1}^{y_2} \{[(p-p_0)\,u]_x + [(p-p_0)\,v]_y\}\,dx\,dy. \tag{2.8}$$

Denoting the $x$-integrals by a bar and assuming either that the disturbance is periodic in $x$ (in which case the bar may be interpreted as implying an average over one wavelength) or that $y$ and $v$ vanish at $x = \pm\infty$ (in which case the bar must be interpreted as implying integration from $x = -\infty$ to $x = +\infty$), we may reduce (2.8) to (cf. Reynolds 1895)

$$\frac{\partial}{\partial t}(T+V) = P + Q, \tag{2.9}$$

where

$$T = \frac{1}{2}\int_{y_1}^{y_2} \rho_0 \overline{(u^2+v^2)}\,dy \tag{2.10}$$

denotes the kinetic energy,

$$V = \tfrac{1}{2} g \int_{y_1}^{y_2} \beta \rho_0 \overline{\eta^2}\,dy \tag{2.11}$$

† Equation (2.4$b$) expresses the fact that a particle at the instantaneous elevation $y$ has the density corresponding to its original elevation $y - \eta$. An alternative, and in some ways more fundamental, approach would be to introduce $\eta$ as the displacement of a streamline in intrinsic co-ordinates *prior* to linearization (cf. Miles, 1959).

the potential energy,

$$P = [-\overline{(p-p_0)\,v}]_{y_1}^{y_2} \tag{2.12}$$

the rate at which work is done on the perturbation flow by the external pressures at the boundaries $y_1$ and $y_2$, and

$$Q = \int_{y_1}^{y_2} \tau U'\, dy \tag{2.13}$$

the rate at which energy is transferred from the mean flow to the perturbation flow by the Reynolds stress

$$\tau = -\rho_0 \overline{uv}. \tag{2.14}$$

We shall restrict the subsequent development to spatially periodic normal modes (see (3.1) below) for which $\beta(y) > 0$. We may regard such motions, in so far as they exist and are either neutrally stable or exponentially unstable, as the dominant components of an asymptotic approximation to the solution of appropriate initial-value problems provided that unstable motions growing more slowly than an exponential cannot exist.† A straightforward extension of the analyses of Eliassen *et al.* (1953) and Case (1960) leads to the conclusion that:

I. The non-exponential growth of a small disturbance in a heterogeneous shear flow cannot be more rapid than $t^{\frac{1}{2}(\nu_m-1)}$, where $\nu_m$ is the maximum value of $[1-4J(y)]^{\frac{1}{2}}$.

It follows that:

II. Dynamic instability of a statically stable, heterogeneous shear flow cannot be other than exponential.

We emphasize, however, that statically unstable flows ($\beta < 0$ in any finite $y$-interval) may, and generally will, exhibit dynamic instabilities of algebraic growth, even though all spatially periodic wave motions are stable. This last statement is amply illustrated by the results of Eliassen *et al.* as described above.

## 3. Periodic motions

We now assume the spatially periodic wave motion

$$\eta(x,y,t) = F(y)\, e^{ik(x-ct)}, \tag{3.1}$$

where $k$ is real, but $c$ may be complex. In accordance with the usual convention, we imply that the imaginary part of the right-hand side of (3.1) is to be discarded in the final reckoning; alternatively, we may regard $F(y;k,kc)$ as a double Fourier transform with respect to $x$ and $t$, the corresponding spectral variables being $k$ and $kc$. Substituting (3.1) in (2.5)–(2.7), we obtain

$$u = -[(U-c)\,\eta]', \quad v = ik(U-c)\,\eta, \quad p-p_0 = \rho_0(U-c)^2\eta', \tag{3.2a, b, c}$$

and

$$[\rho_0(U-c)^2F']' + \rho_0[\beta g - k^2(U-c)^2]\,F = 0. \tag{3.3}$$

The differential equation (3.3) is especially convenient in the study of non-singular oscillations, but we may proceed more directly to the structure of the

† The importance of aperiodic motions in hydrodynamic stability was recognized and developed by Orr (1907), who solved the initial-value problem for small disturbances of an inviscid, plane Couette flow.

general solution by eliminating the first derivative of the dependent variable through the transformation

$$X(y) = \rho_0^{\frac{1}{2}}(y)\,[U(y)-c]\,F(y). \tag{3.4}$$

Substituting (3.4) in (3.3), we may place the resulting differential equation in the standard form

$$X''(y) + h(y)\,X(y) = 0, \tag{3.5}$$

where

$$h = \beta g(U-c)^{-2} - \rho_0^{-1}(\rho_0 U')'\,(U-c)^{-1} - (k^2 + \tfrac{1}{4}\beta^2 - \tfrac{1}{2}\beta'). \tag{3.6}$$

We note that the stream function is given by

$$\psi(x,y,t) = \rho_0^{-\frac{1}{2}}(y)\,X(y)\,e^{ik(x-ct)}, \tag{3.7}$$

and that $X(y)\,e^{ik(x-ct)}$ may be regarded as a stream function if the inertial effects of density variation are neglected, in which case (3.5) reduces to (the usual meteorological approximation)

$$X''(y) + [\beta g(U-c)^{-2} - U''(U-c)^{-1} - k^2]\,X = 0. \tag{3.8}$$

Following Synge (1933), we also may reduce the differential equation for the stream function to standard form through the transformation

$$z = \int_0^y \frac{dy}{\rho_0(y)}, \tag{3.9}$$

under which (3.5) goes over to

$$\frac{d^2\psi}{dz^2} + \rho_0^2\left[\frac{\beta g}{(U-c)^2} - \frac{(\rho_0 U')'}{\rho_0(U-c)} - k^2\right]\psi = 0. \tag{3.10}$$

The point $y = y_c$, defined by

$$U(y_c) = c, \quad U_c' = U'(y_c) \neq 0, \tag{3.11$a,b$}$$

and designated as the *critical layer* for the shear flow, is a regular singularity for each of the foregoing differential equations.† The exponents of this singularity are $\frac{1}{2}(1 \pm \nu)$, where

$$\nu = i\mu = (1-4J_c)^{\frac{1}{2}}, \tag{3.12}$$

and $J_c$ is the Richardson number of (1.1) evaluated at $y = y_c$. Assuming that $\nu$ is not an integer, and that $U(y)$ and $\rho_0(y)$ may be continued analytically into the complex neighbourhood of $y = y_c$, we may apply the method of Frobenius to obtain two, linearly independent solutions to (3.5) in the form

$$X_\pm(y) = (y-y_c)^{\frac{1}{2}(1\pm\nu)}\,w_\pm(y), \tag{3.13}$$

where $w_\pm$ are analytic functions of $y$ in the neighbourhood of $y = y_c$ having the form

$$w_\pm = 1 + \left[(1+J)\frac{(\rho_0 U')'}{\rho_0 U'} - \frac{J\rho_0''}{\rho_0'}\right]_c \frac{(y-y_c)}{1\pm\nu} + \dots. \tag{3.14}$$

† The assumption $U_c' \neq 0$ implies a simple zero of $U-c$ at $y = y_c$. The singularity would be irregular if $U_c' = 0$. We shall not consider this possibility, but it clearly may be of considerable importance for jet-type flows. Indeed, Drazin & Howard (personal communication) have found that the most important long-wave instabilities for jet-type shear flows are those for which $c = U_{max}$, so that $U_c' = 0$.

The Wronskian of these solutions is

$$W\{X_+, X_-\} = X_+X'_- - X'_+X_- = -\nu. \tag{3.15}$$

The solution $X_-$ degenerates if $\nu$ is an integer and, except under very special conditions (see below), must be modified to include a component proportional to $X_+(y)\log(y-y_c)$. The only statically stable flow in this category is that for which $J_c = 0$ $(\nu = 1)$, and the solutions then are essentially similar to those for homogeneous shear flow. We shall not consider them in any detail here, but we remark that the logarithmic component is proportional to the discontinuity in Reynolds stress at $y = y_c$, as calculated in (5.13) and (5.14) below. The singularity at $y = y_c$ is only apparent if this discontinuity vanishes, and both solutions to the differential equation then are regular.

We may regard the singularity at $y = y_c$ as a consequence of either (*a*) our assumption of the periodic wave motion of (3.1), or (*b*) our neglect of diffusion effects; the branch cut at $y = y_c$ can be determined only after either (*a*) posing an initial-value problem and then determining its asymptotic solution as $t \to \infty$, or (*b*) introducing the dominant diffusion effects and then determining the asymptotic solution as the diffusion parameters tend to zero.† The former approach (cf. Eliassen *et al.* 1953 and Case 1960) implies that the path of integration for the inverse transform in the $kc$ spectral plane must pass above $(kc_i > 0)$ the branch point at $kc = kU(y)$ and hence that

$$(U-c) = (c-U)\, e^{\pm i\pi}, \quad k \gtrless 0. \tag{3.16}$$

The latter approach usually has been adopted in considering the stability of homogeneous shear flow (see L 3.4, 3.6, 8.3–8.9), but in the present instance it is necessary to introduce both viscosity and heat conduction to resolve the question. The linearized equations of motion then constitute a system of the sixth order (cf. L 5.2 and 5.3, adding the buoyancy force $(\rho-\rho_0)\,g$ to the $y$-component of the perturbation pressure gradient in L 5.2.2 and L(5.3.9)). We shall not present these equations here, but we remark that a boundary-layer analysis in the neighbourhood of $y = y_c$ leads to the conclusion that the order of magnitude of the actual thickness of the critical layer is given by (cf. L 8.8)

$$\delta = O[(\mu/k\rho_0 U')_c^{\frac{1}{3}}\,(\kappa/\mu R)^{\frac{1}{6}}] \tag{3.17a}$$

$$= O[Re^{-\frac{1}{3}}Pr^{-\frac{1}{6}}(1-\gamma^{-1})^{-\frac{1}{6}}], \tag{3.17b}$$

where $\mu$ denotes the viscosity, $\kappa$ the heat conductivity, $R$ the universal gas constant, $Re$ a Reynolds number based on wavelength, $Pr$ the Prandtl number, and $\gamma$ the specific heat ratio.

Much of the discussion of the next two sections will be concerned with neutral wave motions, for which $c_i = 0$. We designate such motions as singular or non-singular as $y_c$ does or does not lie in the open interval $(y_1, y_2)$. In the former case

† We also might regard the singularity at $y = y_c$ as a consequence of linearization, but it is by no means clear that the behaviour of the linear solution in the neighbourhood of $y = y_c$ could be resolved by a consideration of the non-linear, inviscid periodic flow.

we imply that the limit $kc_i = 0$ must be approached through positive values, and (3.16) implies the transition relations (note that $w_+^* = w_-$ for $J_c > \frac{1}{4}$ and $y$ real)

$$X_\pm^*(y) = X_\pm(y)\, e^{i\pi(1\pm\nu)S(y-y_c)} \quad (J_c < \tfrac{1}{4}), \tag{3.18a}$$

and

$$X_\pm^*(y) = X_\mp(y)\, e^{\pi(i\pm\mu)S(y-y_c)} \quad (J_c > \tfrac{1}{4}), \tag{3.18b}$$

where the asterisk denotes the complex conjugate and

$$S = 0, \quad y > y_c \tag{3.19a}$$

$$= \pm 1, \quad kU_c' \gtrless 0 \quad (y < y_c). \tag{3.19b}$$

We emphasize that (3.19$a$, $b$) are valid only for real $y$.

We conclude this discussion by remarking that the singularity in $X_-$ at $y = y_c$ for $0 < J_c < \frac{1}{4}$ (but not for $J_c = 0$) renders both the kinetic and the potential energies of (2.10) and (2.11) infinite for singular neutral modes. A specific example is provided by Drazin's results for singular neutral modes in the range $\frac{1}{2} < \alpha^2 < 1$, where his solution is essentially $X_-$; for $0 < \alpha^2 < \frac{1}{2}$, on the other hand, his solution is $X_+$, and the energy is bounded. We observe that these infinite energies are a consequence of our too-idealized model and could be removed by including diffusion effects. It also seems likely that these modes would contribute only a finite amount of energy to the solution of an initial-value problem for an ideal fluid provided that the Fourier transform of the initial disturbance were continuous in $k$.

## 4. The eigenvalue problem

We shall consider the eigenvalue problem presented by the assumption of the spatially periodic wave motion (3.1) subject to the boundary conditions

$$\eta_1 = \eta_2 = 0 \quad \text{or} \quad F_1 = F_2 = 0, \tag{4.1a, b}$$

where the subscripts 1 and 2 imply evaluation at $y = y_1$ and $y = y_2$. Many of the subsequent results held for more general boundary conditions, but those of (4.1) hold for the most important cases of horizontal walls at finite values of $y_1$ or $y_2$ and/or null conditions at infinity.

Let $l$ be a characteristic length, $c_*$ a characteristic velocity, and $\beta_*$ a characteristic measure of $\beta$ (we need not introduce an explicit scale for the density $\rho_0$ in virtue of the invariance of the eigenvalue problem with respect to changes in this scale). We then may pose the secular equation for our eigenvalue problem in the form

$$\Delta(c/c_*, \alpha, \lambda, \sigma) = 0, \tag{4.2}$$

where

$$\alpha = kl, \quad \lambda = \beta_* g l^2/c_*^2 \equiv J_*, \quad \sigma = \beta_* l \tag{4.3a, b, c}$$

are dimensionless, real parameters but $c = c_r + ic_i$ may be complex. We may assume $\alpha > 0$ without loss of generality. We define a *neutral surface* as a locus of eigenvalues for which $c_i = 0$ in a $(c_r, \alpha, \lambda, \sigma)$-space. Such a surface will be a *stability boundary* if and only if there exist contiguous eigenvalues for which $c_i > 0$ (unstable wave motions).

We shall assume that $U(y)$ and $\rho_0(y)$ are regular functions of $y$ in $(y_1, y_2)$, that these functions may be continued analytically into a neighbourhood of

$(y_1, y_2)$ that includes the singular point $y = y_c$, and that neither $U'(y)$ nor $\beta(y)$ vanishes in this neighbourhood; on the other hand, we do not exclude the end-points $y_1$ and $y_2$ as possible singularities of the differential equation. These restrictions, especially the requirements $U' \neq 0$ and $\beta \neq 0$, exclude interesting problems (although several of the following theorems, e.g. *V*, obviously can be proved under weaker restrictions), but they guarantee that $c_i$ is a continuous function of the remaining parameters, and hence that the trajectory of a complex eigenvalue must terminate on a stability boundary.

We first remark that

$$F_{\pm} \sim (U-c)^{-1}X_{\pm} \sim (y-y_c)^{\frac{1}{2}(-1\pm\nu)} \quad (y \to y_c), \tag{4.4}$$

in consequence of which the boundary conditions (4.1*b*) imply that:

III. The phase velocity $c$ cannot be equal to $U_1$ or $U_2$.

We observe that this result could not have been inferred from the weaker boundary conditions $v_1 = v_2 = 0$, since $v \sim (U-c)F \sim (y-y_c)^{\frac{1}{2}(1\pm\nu)}$. This paradox is a consequence of linearization, since the approximation $v = ik(U-c)\eta$ is not uniformly valid in the neighbourhood of $U = c$. If $\rho_0'(y_{1,2}) \neq 0$, we may regard the boundary condition $\eta_{1,2} = 0$ as a direct consequence of the physical requirement that a particle at $y = y_{1,2}$ cannot have a density corresponding to an initial position in $y \lessgtr y_{1,2}$ (cf. (2.4*b*)); but if $\rho_0'(y_{1,2}) = 0$ the corresponding boundary condition, and hence III, becomes ambiguous as $c \to U_{1,2}$.

Now let us suppose that $c_i > 0$, so that $U(y) \neq c$ in $(y_1, y_2)$ and $F(y)$ is regular there. Then, multiplying both sides of (3.3) by the complex conjugate $F^*$, integrating between $y_1$ and $y_2$, integrating $[\rho_0(U-c)^2F']'F^*$ by parts, and invoking the boundary conditions (4.1*b*), we obtain (cf. L 8.2)

$$g\int_{y_1}^{y_2} \rho_0\beta|F|^2\,dy = \int_{y_1}^{y_2} \rho_0(U-c)^2(|F'|^2+k^2|F|^2)\,dy. \tag{4.5}$$

Equating the real parts of (4.5), we infer that:

IV. Non-singular neutral modes cannot exist if $\beta(y) < 0$ in $(y_1, y_2)$.

Taking the imaginary part of (4.5), we obtain

$$c_i\int_{y_1}^{y_2} (U-c_r)(|F'|^2+k^2|F|^2)\,dy = 0, \tag{4.6}$$

from which we infer that:

V. The phase velocity $c_r$ for unstable modes ($c_i > 0$) must lie between the maximum and minimum values of $U(y)$ in $[y_1, y_2]$. This implies

$$U_1 < c_r < U_2 \quad \text{if} \quad U' \neq 0 \quad \text{in} \quad (y_1, y_2).$$

We emphasize that this last result is valid for any finite, positive value of $c_i$. If $c_i \to 0^+$ it also may be inferred from VII below, subject to the restriction that $\beta$ and $(\rho_0 U')'$ do not vanish simultaneously. This more restricted form of V was proved originally by Synge (1933) and, independently, by Yih (1957). Drazin (1958*b*) proved V as stated above after neglecting the inertial effects of the density gradient.

A direct corollary of III and V, together with the restrictions guaranteeing the continuity of $c_i$, is:

VI. A stability boundary consists of singular neutral modes—i.e. modes for which $c_i = 0$ and $U(y) = c_r$ in $(y_1, y_2)$.

We remark that the converse of VI is not necessarily true, for we have not demonstrated that unstable modes are contiguous to all singular neutral modes. We also remark that if III were not applicable (as it would not be for homogeneous shear flow), we would have to admit $c = U_1$ and $c = U_2$ as possible end-points for the trajectories of complex eigenvalues and replace $(y_1, y_2)$ by $[y_1, y_2]$ in VI.

## 5. The Reynolds stress

We may deduce further important results for periodic motions in general, and for the eigenvalue problem of the preceding section in particular, from a consideration of the Reynolds stress defined by (2.14). Expressing $u$ and $v$ in terms of $X$ with the aid of (3.1), (3.2*a*, *b*) and (3.4), and substituting the results in (2.14), we obtain (for real $y$)

$$\tau = \tfrac{1}{2}k(X'X^*)_i e^{2kc_i t}, \tag{5.1}$$

where the subscript $i$ implies the imaginary part. Differentiating (5.1) with respect to $y$ and substituting $X''$ from (3.5), we obtain

$$\frac{\partial \tau}{\partial y} = \tfrac{1}{2}k(X''X^*)_i e^{2kc_i t} \tag{5.2a}$$

$$= -\tfrac{1}{2}kh_i |X|^2 e^{2kc_i t} \tag{5.2b}$$

$$= -k^{-1}h_i \rho_0 \overline{v^2}. \tag{5.2c}$$

Referring to (3.6), we obtain

$$h_i = c_i \left[\frac{2\beta g(U-c_r)}{|U-c|^4} - \frac{(\rho_0 U')'}{\rho_0 |U-c|^2}\right]. \tag{5.3}$$

The boundary conditions (4.1) imply

$$\tau_1 = \tau_2 = 0. \tag{5.4}$$

First, let us suppose that $c_i \neq 0$; then, since $\partial\tau/\partial y$ cannot vanish identically, the Reynolds stress must have an extremum and $h_i$ must change sign in $(y_1, y_2)$. This result, which constitutes the extension to heterogeneous shear flows of Rayleigh's theorem on the necessity of a flex ($U'' = 0$) for an unstable mode in a homogeneous shear flow, also follows from the well-known oscillation theorem (Ince 1944, ch. XXI) for the differential equation (3.5) subject to the boundary conditions (5.4). It was proved originally by Synge (1933) and, independently, by Yih (1957) and Drazin (1958*b*). Synge used it to prove that complex values of $c$ must lie on one of the family of circles (with $y$ as the family parameter)

$$\rho_0^{-1}(\rho_0 U')'[(c_r - U)^2 + c_i^2] + 2\beta g(c_r - U) = 0, \tag{5.5}$$

provided that $(\rho_0 U')'$ and $\beta$ do not vanish simultaneously, and hence that

$$|c_i| \leqslant \max. |\beta g \rho_0/(\rho_0 U')'|. \tag{5.6}$$

This result also was given by Yih (1957) except for an erroneous factor of 2.

Synge also proved that complex values of $c$ must be confined to the interiors of the looped curves

$$\tfrac{1}{2}\pi^2\left[\rho_0\int_{y_1}^{y_2}\rho_0^{-1}dy\right]^{-2}[(c_r-U)^2+c_i^2]^2-\rho_0^{-1}(\rho_0 U')'(c_r-U)[(c_r-U)^2+c_i^2] \\ -\beta g[(c_r-U)^2-c_i^2]=0, \quad (5.7)$$

which overlap with the circles of (5.5) in a region that cuts the real axis at $c_r = U_{\min}$ and $c_r = U_{\max}$. This region excludes $c_r = U_{\min}$ and includes $c_r = U_{\max}$ for $c_i > 0$, so that Synge's results confine the minimum value of $c_r$ more severely and its maximum value less severely than V above.

We shall restrict the subsequent discussion of this section to neutral wave motions ($c_i = 0$).† The condition $c_i = 0$ implies, through (5.2) and (5.3), that the Reynolds stress $\tau$ must be constant except for possible discontinuities at $y = y_c$. No such discontinuities can exist for a non-singular motion, in which case (5.4) implies $\tau \equiv 0$. Only one such discontinuity is possible if $U(y)$ is monotonic in $(y_1, y_2)$, and a necessary condition for a singular neutral mode subject to (5.4) is that this discontinuity vanish to yield $\tau \equiv 0$. We conclude that:

VII. The Reynolds stress for any neutral oscillation vanishes identically for monotonic $U(y)$ if $J_c > 0$.

More than one critical layer may exist if $U(y)$ is not monotonic, however, and then the discontinuities in $\tau$ may cancel to satisfy (5.4).

We may calculate rather simple expressions for the Reynolds stress for neutral wave motions by posing a general solution to (3.5) in the form

$$X(y) = AX_+(y)+BX_-(y). \quad (5.8)$$

Substituting (5.8) in (5.1) and setting $c_i = 0+$, we obtain

$$\tau = \tfrac{1}{2}k(|A|^2X'_+X^*_+ + |B|^2X'_-X^*_- + AB^*X'_+X^*_- + A^*BX^*_+X'_-)_i. \quad (5.9)$$

Invoking (3.13) and (3.18$a$, $b$), we find that the first two terms in parentheses are real for $J_c < \frac{1}{4}$ ($\nu$ real), whereas the last two terms are complex conjugates for $J_c > \frac{1}{4}$ ($\nu = i\mu$ imaginary). In the former case, we have

$$X'_+X^*_- = (X'_+X_-)^*e^{-i\pi(1+\nu)S} \quad (5.10a)$$

$$= (\nu+X_+X'_-)^*e^{-i\pi(1+\nu)S} \quad (5.10b)$$

$$= \nu e^{-i\pi(1+\nu)S}+X_+X'^*_-, \quad (5.10c)$$

with the aid of (3.18$a$) and (3.15), and we may reduce (5.9) to

$$\tau = \tfrac{1}{2}k\nu\{AB^*e^{-i\pi(1+\nu)S(y-y_c)}\}_i \quad (J_c < \tfrac{1}{4}). \quad (5.11)$$

Similarly, we obtain

$$\tau = \tfrac{1}{4}k\mu\{|A|^2e^{\pi(i+\mu)S(y-y_c)}-|B|^2e^{\pi(i-\mu)S(y-y_c)}\} \quad (J_c > \tfrac{1}{4}). \quad (5.12)$$

If $J_c = 0$, we must distinguish between $\rho'_{0c} = 0$ (heterogeneous flow with an extremum in the density) and $g = 0$ (heterogeneous flow with negligible buoyancy

† A necessary condition for any neutral motion is, from (2.9), $P+Q = 0$. Calculating $P$ through (3.1) (3.2$b$, $c$), (3.4) and (2.12), we may express this condition in the form $Q = -P = \tau_2 U_2-\tau_1 U_1-c(\tau_2-\tau_1)$. This reduces to $P = Q = 0$ under the boundary conditions (5.4).

force). In the former case we may approximate $\beta$ by $-(\rho_0''/\rho_0 U')_c(U-c)$ and integrate (5.2$c$) between $y_c-$ and $y_c+$ to obtain (cf. L 4.3)

$$\tau(y_c+)-\tau(y_c-) = \frac{\pi}{k}\left[\left(\rho_0 U''+\frac{\rho_0'' g}{|U'|}\right)\frac{\overline{v^2}}{|U'|}\right]_c \quad (\rho_{oc}'=0), \tag{5.13}$$

whereas in the latter case

$$\tau(y_c+)-\tau(y_c-) = \frac{\pi}{k}\left[(\rho_0 U')'\frac{\overline{v^2}}{|U'|}\right]_c \quad (g=0). \tag{5.14}$$

The results (5.11) to (5.14) hold for any neutral wave motion, independently of the boundary conditions at $y_1$ and $y_2$. We shall proceed further on the assumption that $U(y)$ is monotonic ($U' \neq 0$) in $(y_1, y_2)$, so that only a single critical layer can exist and (5.8) to (5.12) remain valid throughout $(y_1, y_2)$ for fixed $A$ and $B$. Imposing the boundary conditions (5.4) on (5.11), we then obtain $(AB^*)_i = 0$ and $AB^*\sin(\pi\nu) = 0$, which imply either $A = 0$ or $B = 0$ (since $\nu$ cannot be an integer), whence:

VIII. A singular neutral mode $X$ for which $0 < J_c < \frac{1}{4}$ must be simply proportional to either $X_+$ or $X_-$.

This theorem, which is illustrated by Drazin's (1958) results, is important in connexion with our previous observation that the modal energy associated with $X_-$ is infinite if $0 < J_c < \frac{1}{4}$.

Imposing (5.4) on (5.12), we obtain $A = B = 0$, whence:

IX. Singular neutral modes cannot exist for monotonic $U(y)$ if $J(y) > \frac{1}{4}$ in $(y_1, y_2)$.

This result was stated by Synge (1933), who asserted that it follows directly from the fact that neither $X_+$ nor $X_-$ can be real. This argument appears to overlook the fact that $X$, as given by (5.8), is real if $B = A^*$, in consequence of which IX can follow only after the invocation of appropriate boundary conditions.† We have not, of course, ruled out non-singular neutral modes for $J(y) > \frac{1}{4}$ in $(y_1, y_2)$.

Now let us consider the possibility of unstable modes for $J(y) > \frac{1}{4}$ in $(y_1, y_2)$. We can approach this question by considering the asymptotic behaviour of $c$ as a function of the parameter $\lambda$ (see (4.2) and (4.3)), say $c(\lambda)$. We assume that $J(y)$ is positive definite and of prescribed functional form, but that $\lambda$ may be varied. (It is perhaps simplest, conceptually, to imagine $\beta(y)$ and $U'(y)$ to be prescribed and $g$ varied, but we also may prescribe $\beta(y)/\beta_*$ and $U'(y)\,l/c_*$ and vary any or all of $\beta_*$, $c_*$, $l$ and $g$.) Let $\lambda_0$ be the minimum value of $\lambda$ for which $J(y) \geqslant \frac{1}{4}$ in $(y_1, y_2)$. Then, from IX, $c(\lambda)$ must either be definitely complex ($|c_i| > 0$) or non-singular (real and not in $[U_1, U_2]$) for $\lambda > \lambda_0$. Applying Liouville's method to (3.5), we obtain the asymptotic solutions

$$X_\pm \sim (\beta g)^{-\frac{1}{4}}(U-c)^{\frac{1}{2}}\exp\left[\pm i\int_{y_1}^{y}(\beta g)^{\frac{1}{2}}(U-c)^{-1}dy\right] \quad (\lambda\to\infty). \tag{5.15}$$

Invoking the boundary conditions $X_1 = X_2 = 0$, we obtain

$$\sin\left[\int_{y_1}^{y_2}\frac{(\beta g)^{\frac{1}{2}}dy}{U-c}\right] = 0 \quad (\lambda\to\infty), \tag{5.16}$$

† Prof. Synge (private communication) agrees with this statement.

as the asymptotic form of the secular equation. Assuming $U' \neq 0$ in $(y_1, y_2)$, we see that (5.16) can be satisfied only if $c$ is real and not in $[U_1, U_2]$, corresponding to a non-singular neutral mode. It follows that $c_i(\lambda) \equiv 0$ for $\lambda > \lambda_0$ and hence that:

X. Sufficient conditions for the stability of a heterogeneous shear flow are $U'(y) \neq 0$ and $J(y) > \frac{1}{4}$ in $(y_1, y_2)$.

## 6. Concluding remarks

The most important questions left unresolved by the preceding analysis appear to concern (*a*) the conditions under which a locus of singular neutral modes constitutes a true stability boundary, and (*b*) the significance of singular neutral modes having infinite energies. It appears unlikely that (*a*) could be resolved satisfactorily without first resolving (*b*).

It also would be desirable to give further consideration to non-montonic velocity distributions, but this would appear to raise some difficult mathematical questions.

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