A fully analytical approach to the problem of tsunami generation by nonlinear acoustic mechanism

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Abstract

The origin of elastic oscillations of water column in a tsunami source is due to the excitation of standing acoustic waves in the natural quarter-wave resonator of a "compressible liquid with a free surface on the rigid bottom". The resonator exhibits a set of frequencies (normal modes): $c(1+2k)H/4$, where c is the sound velocity in water, H is the ocean depth, and k = 0,1,2,3,.... Energy of the elastic oscillations is much larger than energy of tsunami wave (initial elevation): Wac/Wts=2c/g>>1. Due to non-linear energy transfer, the elastic oscillations may contribute the long gravitational waves (tsunamis). In this study we suggest an analytical approach to the problem of mathematical description of tsunami generation by the nonlinear mechanism. We consider 2D problem in a vertical plane 0xz. Mathematical model is based on the nonlinear Euler equations where we assume all the fluid fields (velocity, pressure, density) to consist of fast-oscillating and time-averaged (slow) terms. Then, the equations are averaged in time. The non-linearity of the Euler equations introduces additional terms in the time-averaged flow equations. These additional terms can be interpreted as external mass force and distributed mass source; we consider them as a non-linear tsunami source. Linearized time-averaged flow equations are reduced to heterogeneous wave equation. The elastic oscillations of water column are expressed as a sum of normal modes. We assume that due to the refraction of the hydroacoustic waves into the underlying elastic half-space amplitude of the normal modes exponentially decrease in time. Finally, we estimate contribution of the nonlinear mechanism in tsunami wave.

Basic mathematical model

$$\left.\begin{aligned}\vec v &= \langle \vec v\rangle + \vec v' \\ p &= \langle p\rangle + p' \\ \rho &= \langle \rho\rangle + \rho'\end{aligned}\right\} \Rightarrow \begin{cases}\dfrac{\partial \vec v}{\partial t} + (\vec v,\vec\nabla)\vec v = -\dfrac{\vec\nabla p}{\rho} + \vec g \\ \dfrac{\partial \rho}{\partial t} + \mathrm{div}(\rho\vec v) = 0\end{cases} \Rightarrow \begin{cases}\dfrac{\partial\langle\vec v\rangle}{\partial t} = -\dfrac{\vec\nabla\langle p\rangle}{\rho_0} + \vec g + \vec\Phi \\ \mathrm{div}(\langle\vec v\rangle) = s\end{cases} \Rightarrow$$

non-linear tsunami source

Generation of long gravitational waves by the non-linear tsunami source

$$\frac{\partial^2\xi}{\partial x^2} - \frac{1}{gH}\frac{\partial^2\xi}{\partial t^2} = \frac{1}{gH}Q(x,t)$$

$$Q(x,t) = \int_{-H}^{0} dz\left[\frac{\partial\Phi_x}{\partial x} + \int_{z}^{0}\frac{\partial^2\Phi_z}{\partial x^2}dz^* - \frac{\partial s}{\partial t}\right]$$

where

$$\vec\Phi = -\langle(\vec v',\vec\nabla)\vec v'\rangle + \frac{1}{2c^2\langle\rho\rangle^2}\langle\vec\nabla p'^2\rangle$$

$$s = -\frac{1}{c^2\langle\rho\rangle}\mathrm{div}\langle p'\vec v'\rangle$$

Auxiliary problem

$$\frac{\partial^2 F}{\partial t^2} - c^2\frac{\partial^2 F}{\partial z^2} = 0$$

$$F = 0, \quad z = 0$$

$$\frac{\partial F}{\partial z} = V(t), \quad z = -H$$

$$V(t) = \frac{\eta_0}{\tau}e^{-t/\tau}$$

$$p' = -\rho\frac{\partial F}{\partial t}, \quad v' = \frac{\partial F}{\partial z}$$

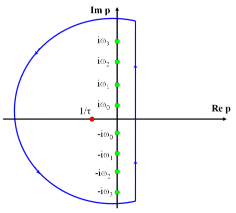
Analytical solution of the auxiliary problem

$$z^* = z/H, \quad t^* = tc/H, \quad \tau^* = \tau c/H, \quad p^* = pH/c$$

$$F(z,t) = V_0\frac{H}{2\pi i}\int_{s-i\infty}^{s+i\infty}dp^*\,\frac{\tau^* e^{p^*t^*}}{(1+p^*\tau^*)}\frac{\mathrm{sh}\,p^*z^*}{p^*\,\mathrm{ch}\,p^*}$$

$$\frac{F(z,t)}{V_0H} = \frac{e^{-t/\tau}\,\tau\,\mathrm{sh}\,z/\tau}{\mathrm{ch}\,1/\tau} + \sum_{n=0}^{\infty}\frac{4(-1)^n\tau}{\pi(1+2n)}\sin\left(\frac{\pi z}{2}(1+2n)\right)\times$$

$$\times\frac{\sin\left(\frac{\pi t}{2}(1+2n)\right) - \frac{\pi\tau}{2}(1+2n)\cos\left(\frac{\pi t}{2}(1+2n)\right)}{1+\left(\frac{\pi\tau}{2}\right)^2(1+2n)^2}$$



Contribution of nonlinearity in tsunami amplitude

$$\Phi_z = \int_{-H}^{0}\int_{z}^{0}dz\,d\hat z\left\langle\frac{1}{c^2}\frac{\partial F}{\partial t}\frac{\partial^2 F}{\partial t\,\partial z} - \frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z^2}\right\rangle =$$

$$= -V_0^2H\sum_{n=0}^{\infty}\frac{\tau^2}{\left(1+\left(\frac{\pi}{2}\tau(1+2n)\right)\right)^2}$$

$$\Phi_z(x,t) = \Phi_z e^{-x^2/L^2}e^{-t/T}$$

$$T = D\frac{H}{c}, \quad D = \frac{(1+\rho_b V_p/\rho c)^2}{2\rho_b V_p/\rho c}$$

$$D \approx 7 - 9.5$$

$$\xi^*(x^*,t^*) = A_0\int_0^{t^*}\frac{e^{-t^*/T^*}}{T^*L^*}\left[x^* e^{-2\hat x^2/L^{*2}}\right]_{x^*-(t^*-\hat t)}^{x^*+(t-\hat t)}d\hat t$$

$$A_0 = \frac{2\eta_0^2 cD}{\sqrt{gH}\,L}\sum_{n=0}^{\infty}\frac{1}{\left(1+\left(\frac{\pi}{2}\frac{\tau c}{H}(1+2n)\right)\right)^2}$$

$$x^* = x/H, \quad t^* = t\sqrt{g/H}$$

Wave profile generated by nonlinear tsunami source