Явные равномерные асимптотики для линейных волн на воде со слабой и сильной дисперсиями.

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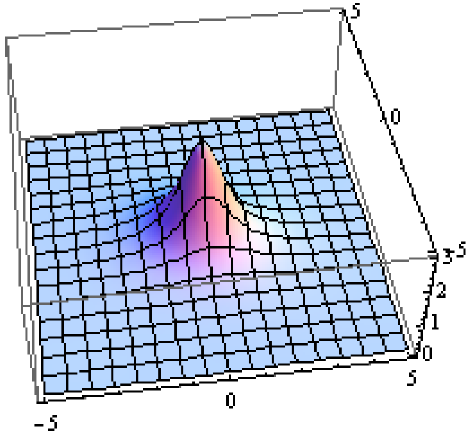
FORMULATION OF THE PROBLEM

The linear water wave equations in dimensionless variables $x = (x_1, x_2), y$ **for the potential** $\Phi(x, y, t)$ **in the basin with the slow varying bottom** $D(x)$:

$$h^2 \triangle \Phi + \Phi_y y = 0, \quad -D(x) < y < 0, x \in \mathbb{R}^2_x$$
$$\Phi_y + h^2 \langle \nabla D, \nabla \Phi \rangle = 0, \quad y = -D(x), \quad h^2 \Phi_{tt} + \Phi_y = 0, \quad y = 0.$$

Here h is the small parameter, characterizing the slow varying bottom.



The main unknown object: the elevation of free surface

$\eta(x, t) = -h \frac{\partial \Phi}{\partial t}|_{y=0}$

The Cauchy-Poisson problem (piston model)

$$\Phi|_{y=0,t=0} = 0, \quad \frac{\partial \Phi}{\partial t}|_{y=0,t=0} \equiv -\eta|_{t=0} \equiv -\eta^0 = -V(\frac{x - x^0}{\mu}), \quad x^0 = 0$$

$V(y)$ is a finite or rapidly decaying function as $|y| \to \infty$,
μ is the *second small parameter*, characterizing the size of the initial perturbation $1 \gg \mu \gg Ch$, $C > 0$.

The reduction to the Cauchy problem for the pseudodifferential equation on the surface $y = 0$
(S.Dobrokhotov,1983, S.Dobrokhotov, P.Zhevandrov 1985)

$$h^2\frac{\partial^2\eta}{\partial t^2}+\hat{\mathcal{H}}\eta = 0, \quad \hat{\mathcal{H}} = \mathcal{H}\left(\overset{2}{x}, -ih\overset{1}{\frac{\partial}{\partial x}}, h\right), \qquad \eta|_{t=0} = V\left(\frac{x}{\mu}\right), \qquad \eta_t|_{t=0} = 0,$$

$$\mathcal{H}(x,p,h) = H(x,p) - \frac{ih}{2}\sum_{j=1}^{2}\frac{\partial^2\mathcal{H}_0}{\partial x_j\partial p_j}(x,p) + O(h^2), \qquad \mathcal{H}_0 = |p|\tanh\big(D(x)|p|\big).$$

The formal answer $\eta = \cos\left(\frac{t}{h}\sqrt{\hat{\mathcal{H}}}\right)V\left(\frac{x}{\mu}\right)$.

The construction of asymptotics: the passage to the Maslov canonical operator (Fourier integral operators) leads to the symbols

$$H_\pm = \pm H(p,x) \equiv \pm\sqrt{|p|\tanh\big(D(x)|p|\big)} = \pm|p|\sqrt{D(x)} + O(|p|^3). \quad \Longrightarrow$$

One has nonsmooth intersection of characteristics when $p = 0$.

Geometry of the rays and trajectories in $2-D$ phase space $\mathbb{R}^4_{p,x}$:
Regular and singular Lagrangian manifolds, standard and nonstandard caustics, leading front edges

$$\dot{x} = H_p, \quad \dot{p} = -H_x, \quad p_{t=0} = \alpha \in \mathbb{R}^2, \qquad x_{t=0} = x^0$$

The Lagrangian manifolds:

$$\Lambda_t = \{p = P(t,\alpha), x = X(t,\alpha)\} = g^t_H\{p = \alpha, x = x^0\}$$

Standard caustics: $J = \det \frac{\partial X}{\partial \alpha} = 0$

Nonstandard caustics (leading front edges) are **singularities** of Λ_t

Strong singularities $J|_{\Lambda_t} \equiv 0$

Small momenta p correspond to long waves which
1) have the biggest velocity of propagation and
2) organize the most interesting object of the problem: the head wave and the leading front edge.

The dispersion effects

In the water wave case $H=\sqrt{|p|\tanh(D(x)|p|)}$ is not smooth at $p=0 \quad \Longleftrightarrow \quad \rho=0$!

Nevertheless we can construct Λ_t and study it behavior for small ρ. Easy to prove that $\rho \to +0$ corresponds to maximum velocities along the trajectories (P, X). We have

$$H(x,p)=H^0(x,p)+H^1(x,p)+O(|p|^5),$$

$$H^0(x,p)=|p|\sqrt{D(x)},\; H^1(x,p)=-\frac{1}{6}(D(x))^{5/2}|p|^3.$$

$$P=\rho P^0(t,\phi)+\rho^3 P^1(t,\phi)+O(\rho^5), \quad X=X^0(t,\phi)+\rho^2 X^1(t,\phi)+O(\rho^4).$$

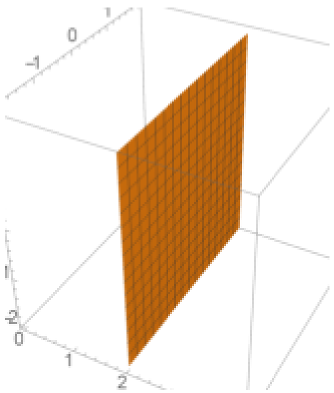
The vector- functions $(P^0(t,\phi), X^0(t,\phi))$ give the trajectories of the "limit" Hamilton system with Hamiltonian $H^0(x,p)$ corresponding to the limit wave equation $u_{tt}=\mathrm{div}(D(x)\nabla u)$.

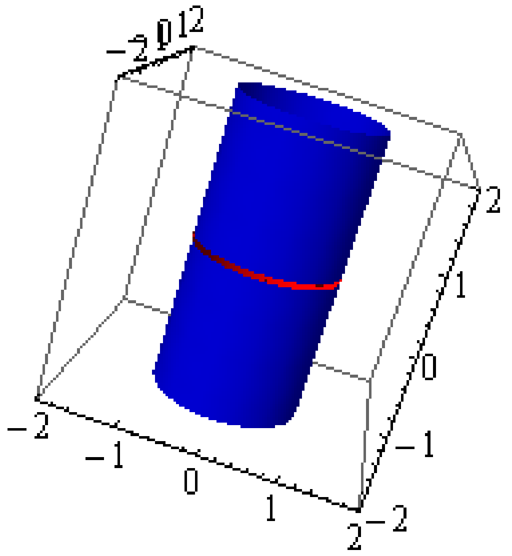
Thus
1) the projection Λ_t on $\mathbb{R}^2_x$ organizes the compact set bounded by the **leading edge front** $\gamma_t=\{x\in \mathbf{R}^2\colon x=X^0(t,\phi)\}$.
2) The manifolds Λ_t has singularities for $\rho=0$ which are its boundary. We puncture a small neighborhood of the point $p=0$ $(\rho=0)$ and obtain **the punctured Lagrangian manifold**.

The temporal evolution of the Lagrangian manifolds

in the case of the constant bottom

$t = 0$



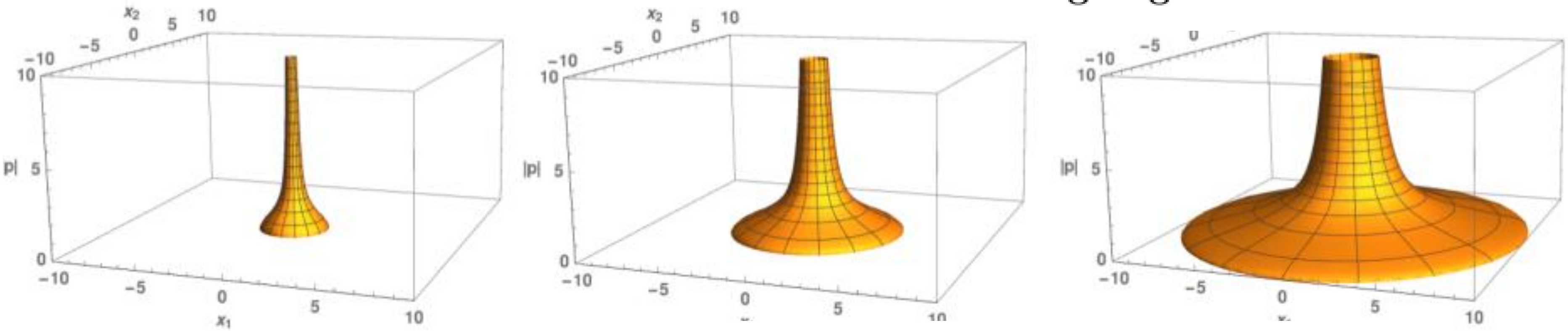


Dispersionless and small dispersion cases

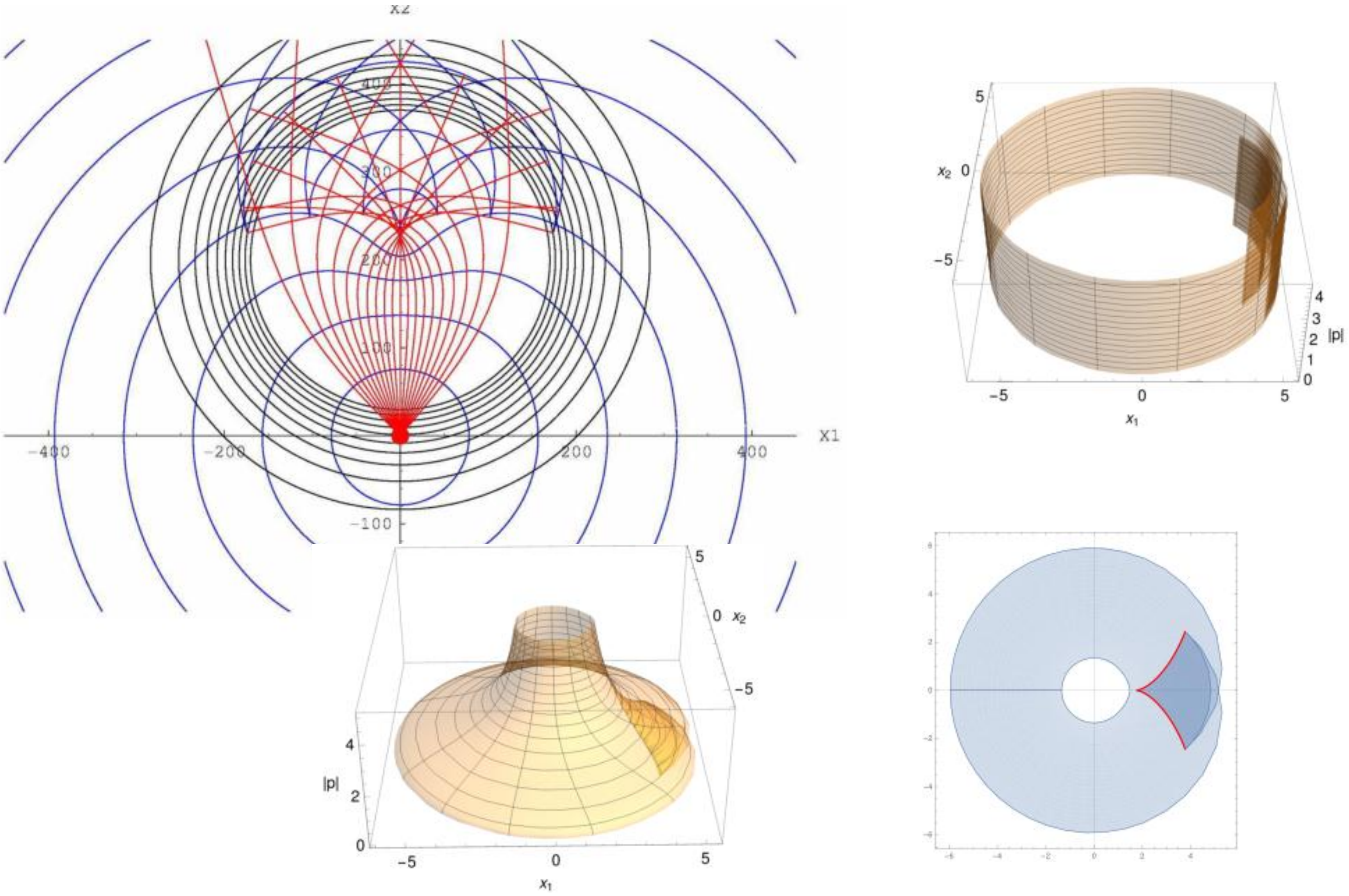
The strong singular Lagrangian manifold

The singular Lagrangian manifold
with nonstandard caustic *"in general position"*
= leading edge front

Strong dispersion case



"punctured" Lagrangian manifold


X2
X1
-400
-200
200
400
100
-100
x2
x1
|p|

Dispersionless case

The wave field outside the focal points

$$\eta \approx \sqrt{\frac{\mu}{|X_\psi(\psi,t)|}}\sqrt[4]{\frac{D(x_0)}{D(x\ )}}\mathrm{Re}\Big[\mathrm{e}^{-\mathrm{i}\pi\mathrm{m}(\psi,\mathrm{t})/2}\mathrm{F}\Big(\frac{\mathrm{y(x,t)}}{\mu}\sqrt{\frac{\mathrm{D(x_0)}}{\mathrm{D(x\_)}}},\psi\Big)\Big|_{\psi=\psi(\mathrm{x,t})}\Big]$$

Here

$y(x,t)$ is the alternative distance between the point x and the closest point $X(\psi(x,t),t)$ on the front,

$\psi(x,t)$ is the correspondence angle (coordinate) on the front,

$m((\psi,t))$ is the Morse (Maslov) index of this point.

$$F(s,\psi)=\frac{e^{-i\pi/4}}{\sqrt{2\pi}}\int_0^\infty \widetilde{\eta}^{\,0}(\rho\mathbf{n}(\psi))\sqrt{\rho}e^{is\rho}\,d\rho, \qquad \widetilde{\eta}^{\,0}(k)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\eta^0(z)e^{i\langle k,z\rangle}\,dz\ , \quad \mathbf{n}(\psi)=\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}$$

$\int_0^\infty(\cdot)d\rho$ gives the passage from fast oscillating functions to fast decaying ones

Asymmetric Dotsenko-Sergievskii-Cherkasov-Wang type source

$$\eta^0(z)=\frac{A}{(1+(z_1/B_1)^2+(z_2/B_2)^2)^{3/2}}, \qquad F(s,\psi)=\frac{Ae^{-i\frac{\pi}{4}}}{2\sqrt{2}\left(\sqrt{B_1^2\cos^2\psi+B_2^2\sin^2\psi}-is\right)^{3/2}}.$$

Caustics: 1) standard $X_\psi=0$, 2) shore D(x)=0

$c(x)=0$

Asymptotics near the leading front edge ("nonstandard caustic") outside of the focal points

The asymptotic formulas in the neighborhood of the regular points of the leading edge front $(X^0_\phi(\phi,t) \neq 0)$:

$$\eta_{as} = \frac{\sqrt{\mu}}{\sqrt{|X^0_\phi|}} \sqrt[4]{\frac{D(0)}{D(X^0(t,\phi(t,x)))}} \mathrm{Re}\Big[\Big(\frac{e^{-i\frac{\pi}{4}-i\frac{\pi m}{2}}}{\sqrt{2\pi}}$$

$$\times \int_0^\infty \sqrt{\rho} e^{\frac{i}{\mu}(\rho\Delta(t,x)-\delta^2\rho^3\Theta(t,x,\phi(x,t)))} \widetilde{V}\big(\rho\mathbf{n}(\phi(x,t))\big)d\rho\Big)\Big],$$

$$\Theta(t,x,\phi) = \frac{D^{3/2}(0)}{6}\int_0^t D(X^0(\tau,\phi))d\tau,$$

here $\phi_0 = \phi(t,x)$ is the solution to the equation $\langle X^0_\phi(t,\phi), x - X^0(t,\phi)\rangle = 0$.
It is *the simplest uniform* formula for $1 \gg \mu \geq h > 0$.

S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. A. Tolchennikov, Uniform Formulas for the Asymptotic Solution of a Linear Pseudodifferential Equation Describing Water Waves Generated by a Localized Source, Russian Journal of of Mathematical Physics, Vol. 27, No. 2, 2020, pp. 185-191

Dispersionless and small dispersion cases

Important example (Sretenskii, Dotsenko, Sergievskii, Cherkasov, La Mohita, Wang)

$$V(y) = V^0(T(\theta)y), \quad V^0(y) = \frac{A}{\left(1 + \frac{y_1^2}{b_1^2} + \frac{y_2^2}{b_2^2}\right)^{\frac{3}{2}}}, \quad T(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

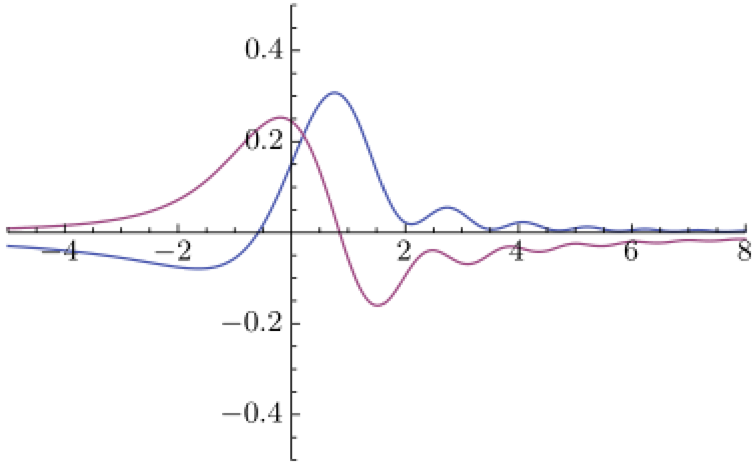
$$\widetilde{V}(\rho\mathbf{n}(\phi) = Ab_1b_2e^{-\rho\sigma(\phi,\theta)}, \qquad \sigma = \sqrt{b_1^2\cos^2(\phi-\theta) + b_2^2\sin^2(\phi-\theta)},$$

Then

$$\eta_{as} = \frac{Ab_1b_2\sqrt{\mu^3}}{h\sqrt{|X^0_\phi(t,\phi)|}}\sqrt[4]{\frac{D(0)}{D(X^0(t,\phi)}}\frac{\mathrm{Re}\left[e^{-i\frac{\pi m}{2}}\mathbf{U}(z)\right]}{\sqrt{\Theta(x,t,\phi)}}\bigg|_{\phi_0=\phi(x,t)},$$

$$z = \frac{\Delta(x,t) + i\mu\sigma(\phi_0,\theta)}{2h^{2/3}\sqrt[3]{\Theta(x,t,\phi_0)}}\bigg|_{\phi=\phi(x,t)}$$

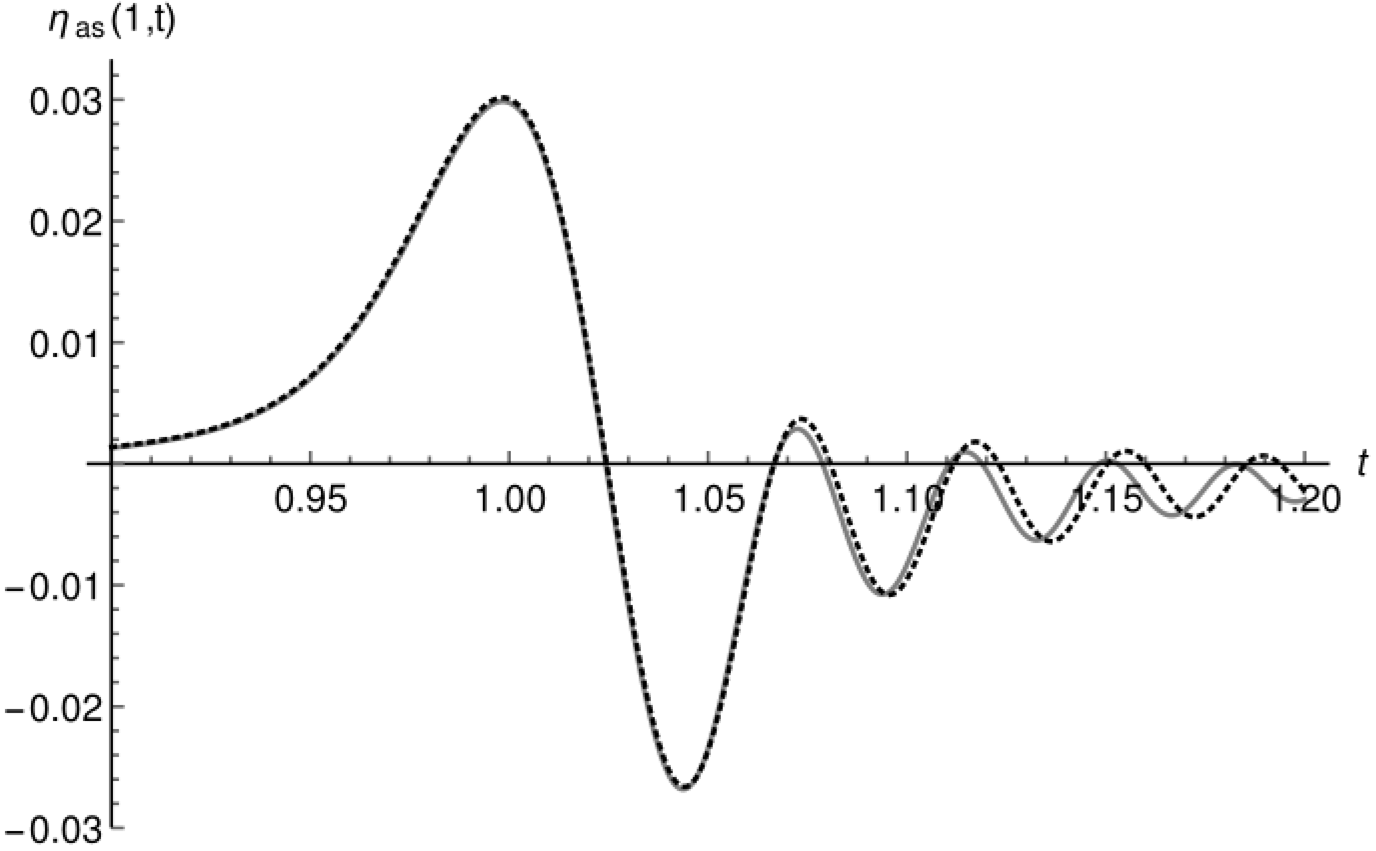
$$\mathbf{U}(z) = -\frac{\pi}{\sqrt{6}}\frac{d(Ai^2(z) + iAi(z)Bi(z}{dz}$$



Here Ai, Bi are Airy functions.

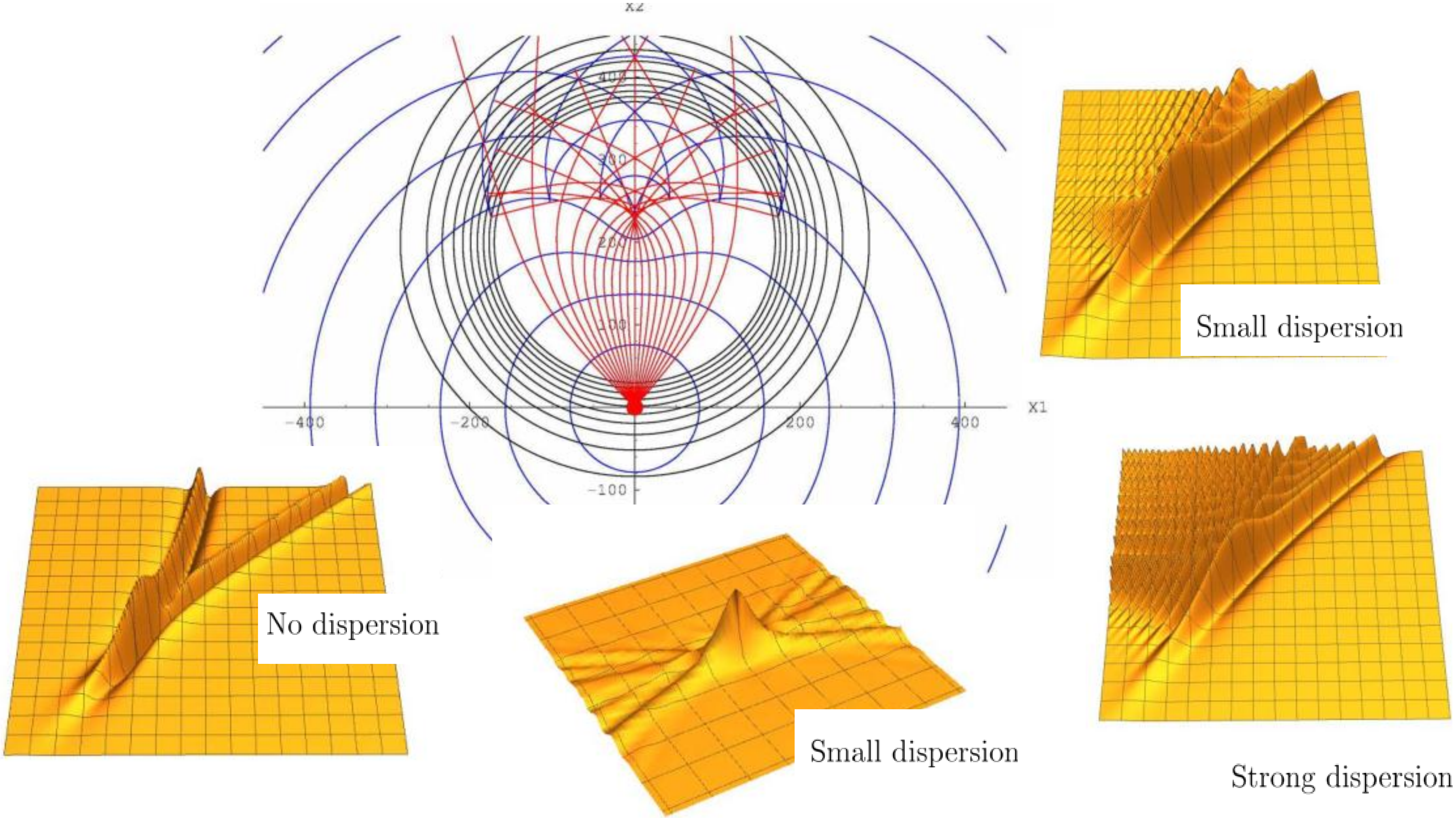
Weak dispersion : $h^2 \sim \mu^3$

$\mu = 0,02, h = \mu^{\frac{3}{2}} \approx 0,0028$ $\qquad$ $\delta \approx 0,14, l \approx 28\,km, L \approx 1414\,km$



grey line: the exact solution

small dashed line: the asymptotics based on the Airy function



X2
X1
-400
-200
200
400
400
300
200
100
-100
Small dispersion
No dispersion
Small dispersion
Strong dispersion

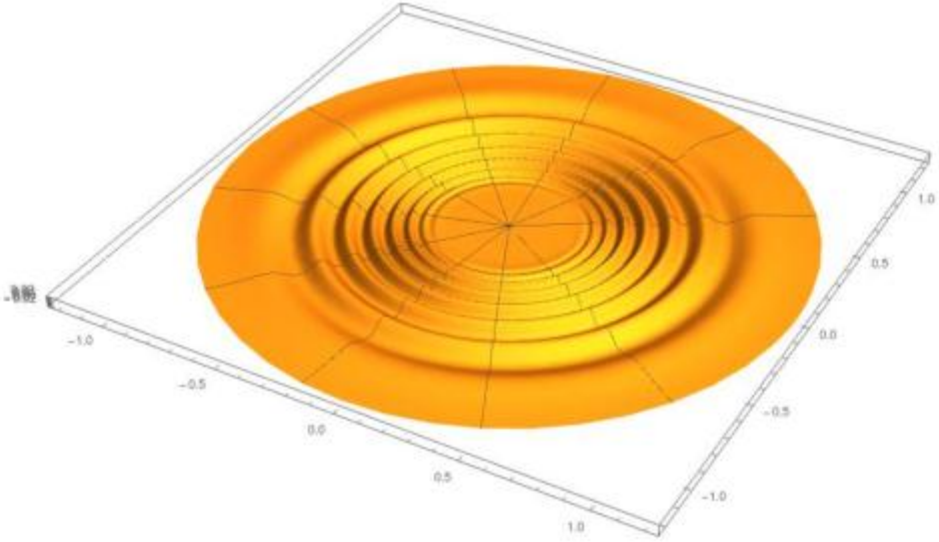
The uniform asymptotics in wide neighborhood of the leading front edge outside of focal points. The global asymptotics in the case without standard caustics.

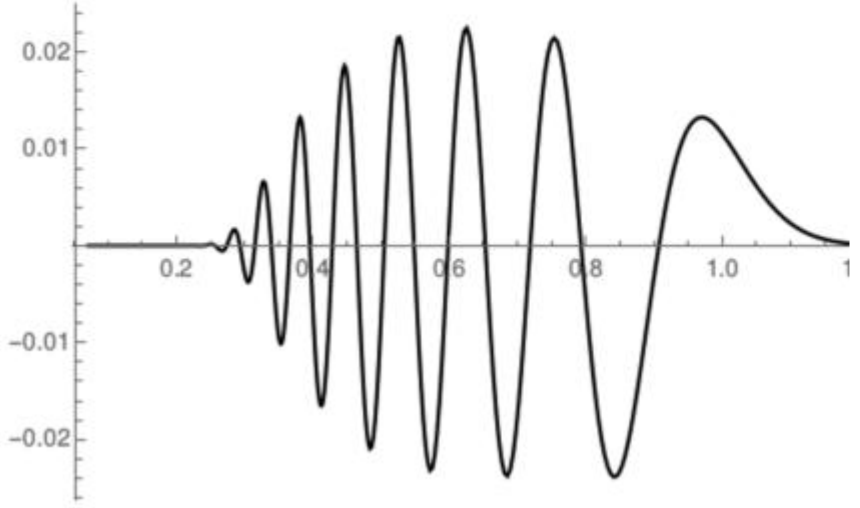
This formula works for any initial perturbation!

$$\eta_{as}(t,x) \sim -\pi h \sum_{i=1}^{k} \mathrm{Re}\left[e^{-\frac{i\pi m_i}{2}} \frac{A'(-(-\frac{3S(\alpha_i)}{4h})^{\frac{2}{3}})\widetilde{V}(\alpha_i)}{|J(\alpha_i)|^{\frac{1}{2}}} \right]$$

$$A(z) = Ai(z)^2 + i\,Ai(z)Bi(z),$$
$$S(t,\alpha) := \int_0^t (\langle p, H_p\rangle - H)|_{x=X(t,\alpha),p=P(t,\alpha)} dt$$

Strong dispersion: $h \sim \mu$





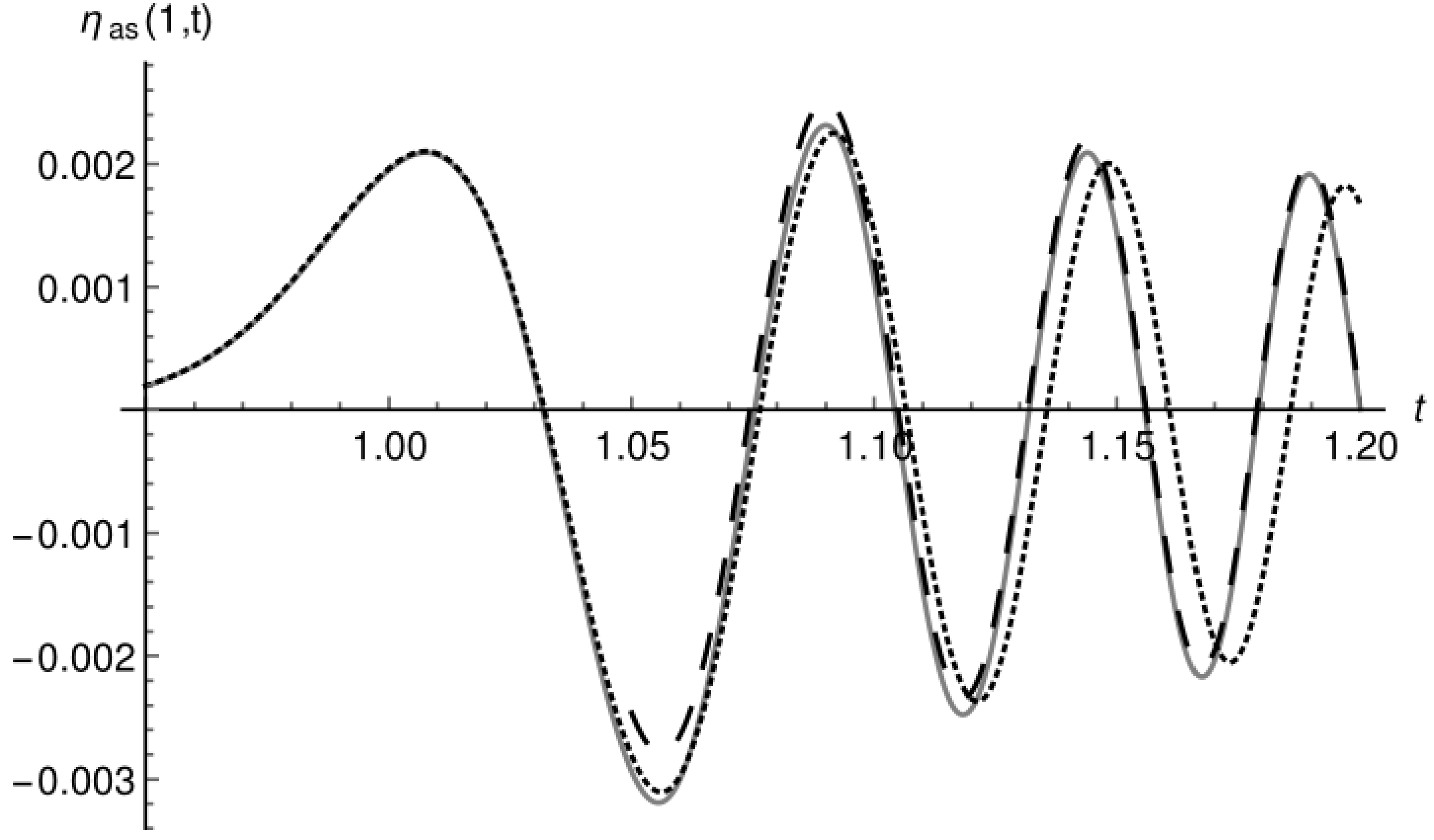
Important conclusion:

1) The wave field near caustics is usually described using parametrically defined functions.

2) A very good description of such functions is given using Lagrangian manifolds, the parameters are coordinates on these manifolds.

Strong dispersion: $h \sim \mu$

$\mu = h = 0,004 \quad \delta = 1 \qquad l = 4\,km, L = 1000\,km$



grey line: the exact solution
small dashed line: the asymptotics based on the Airy function
large dashed line: the WKB asymptotics

ANSATZes

in the vicinity of simple standard and nonstandard caustics

$g_1\mathrm{Ai}(z)+g_2\mathrm{Ai}'(z)$ $\quad$ $g_1\mathrm{J}_0(\sqrt{-z})+g_2\mathrm{J}_0{}'(\sqrt{-z})$ $\quad$ $g_1\frac{d\mathrm{Ai}^2}{dz}(z)+g_2\frac{d(\mathrm{AiBi})}{dz}(z)$

Airy **Bessel** **Airy²**

Oscillating (WKB-type) anzatz's (known) asymptotics ($z << -1$)

$$(b_1^+(z)g_1 + b_1^+(z)g_2))e^{i\mathcal{F}(z)} + (b_1^-(z)g_1 + b_1^-(z)g_2))e^{-i\mathcal{F}(z)}$$

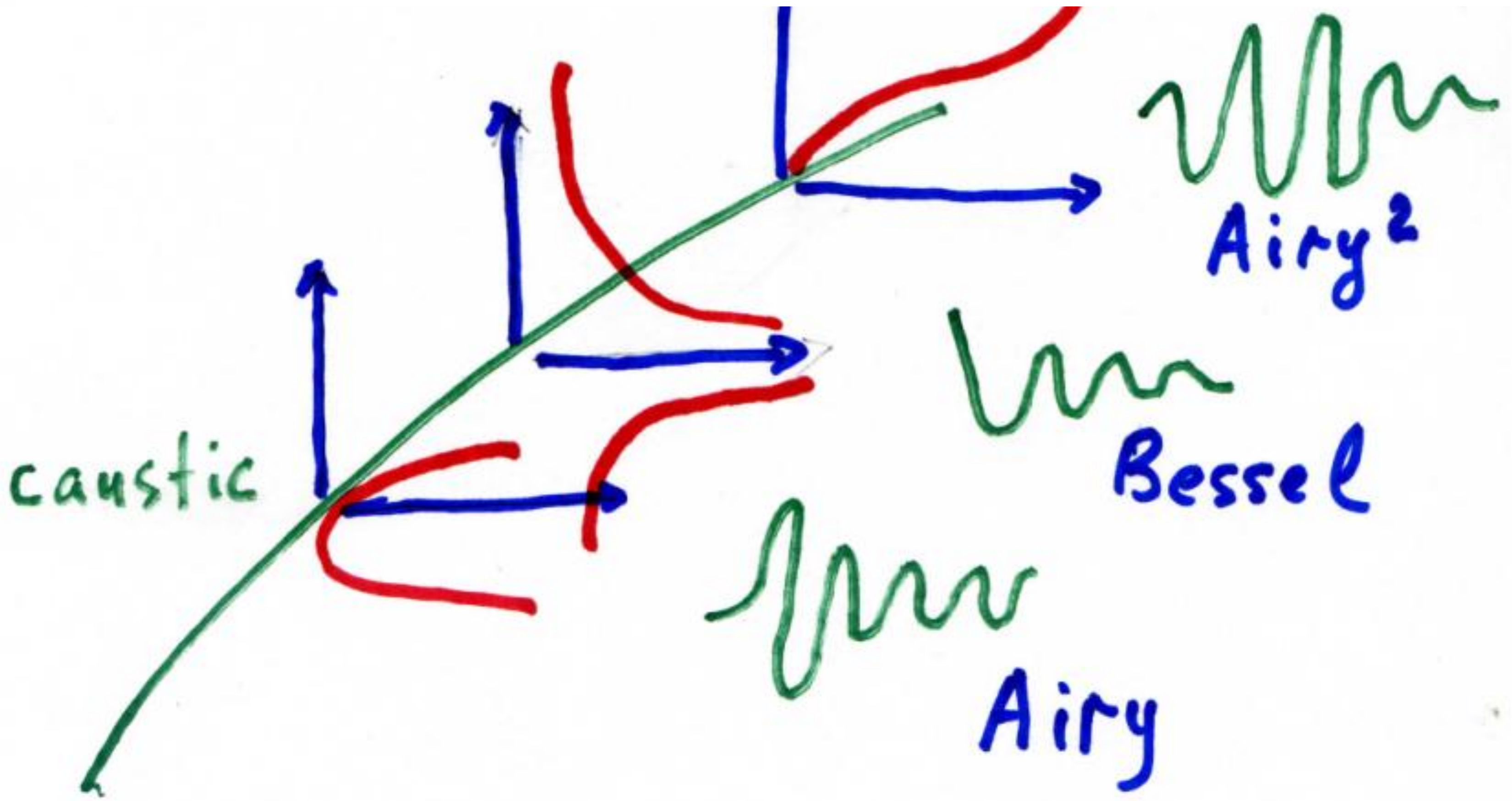
Direct WKB (known) asymptotics: $a_+(x,t)e^{\frac{S_+(x,t)}{h}}+a_-(x,t)e^{-\frac{S_-(x,t)}{h}}$

$$= e^{\frac{S_+(x,t)+S_-(x,t)}{h}}\Big(a_+(x,t)e^{\frac{S_+(x,t)-S_-(x,t)}{h}}+a_-(x,t)e^{-\frac{S_+(x,t)-S_-(x,t)}{h}}\Big), x \in \mathbb{R}^n$$

We choose $z = Z(x,t,h) = \mathcal{F}^{-1}(\frac{S_+(x,t)-S_-(x,t)}{h})$ and $g_j(x,t,h)$:

$$b_1^+(Z)g_1 + b_1^+(Z)g_2 = a_+(x,t) \quad b_1^-(Z)g_1 + b_1^-(Z)g_2 = a_-(x,t)$$

Simple standard or non standard caustic (fold)



Благодарю за внимание!