Асимптотики длинных нелинейных волн, захваченных берегами

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Nonlinear shallow water equations

$$\frac{\partial \boldsymbol{\eta}}{\partial t} + div((\boldsymbol{\eta} + D(x))\mathbf{u}) = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}, \nabla)\mathbf{u} + g\nabla\boldsymbol{\eta} = 0, \quad x \in \mathbb{R}^2_x$$

$\mathbf{u}(x,t)$ is the 2-velocity, $\boldsymbol{\eta}(x,t)$ is the free elevation

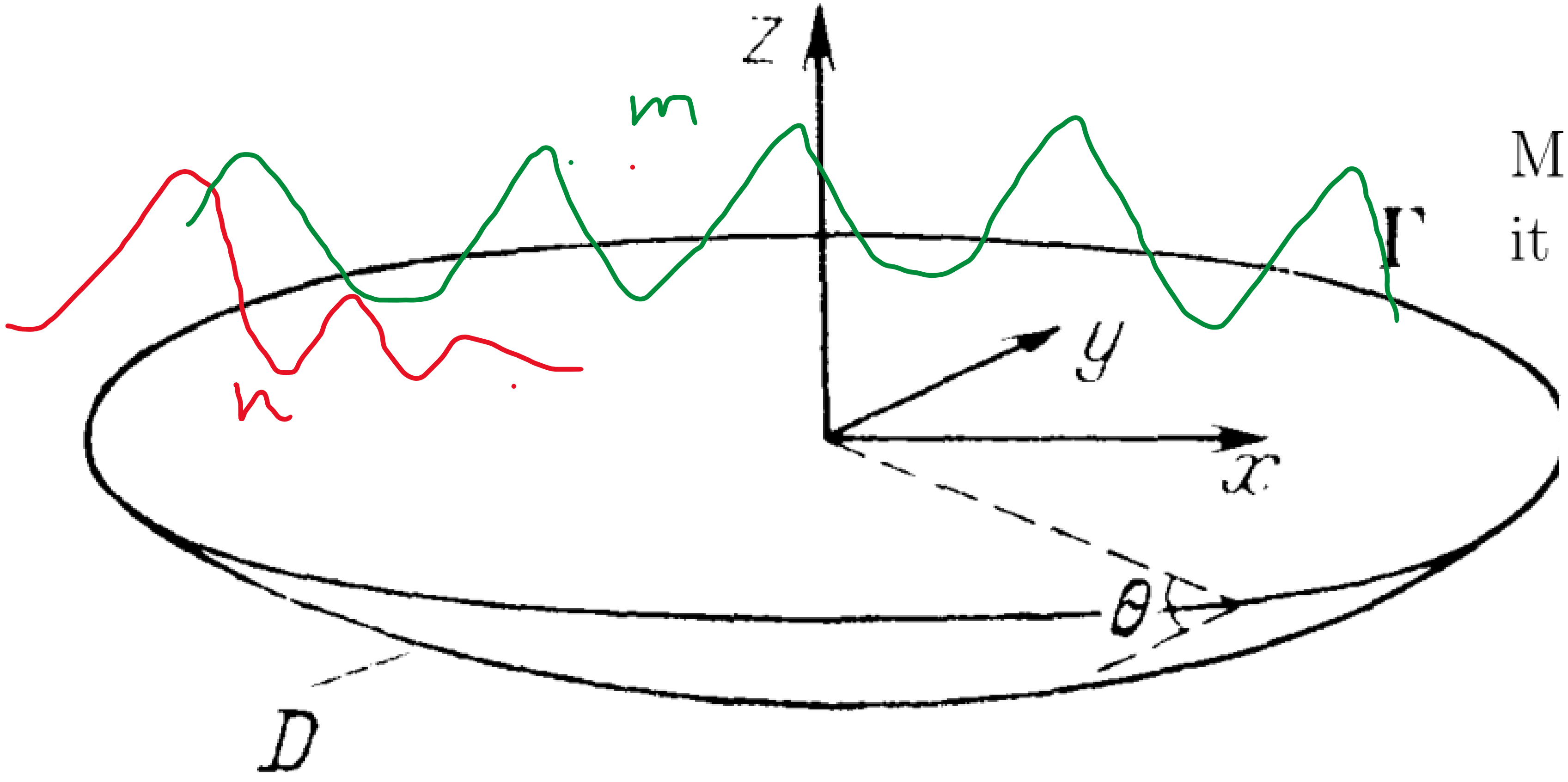
The smooth function $D(x)$ describes the depth of the basin,

$\partial\Omega = \{D(x) = 0\}$ is the basin's shoreline, $\frac{\partial D}{\partial \mathbf{n}}|_{\partial\Omega} \neq 0$

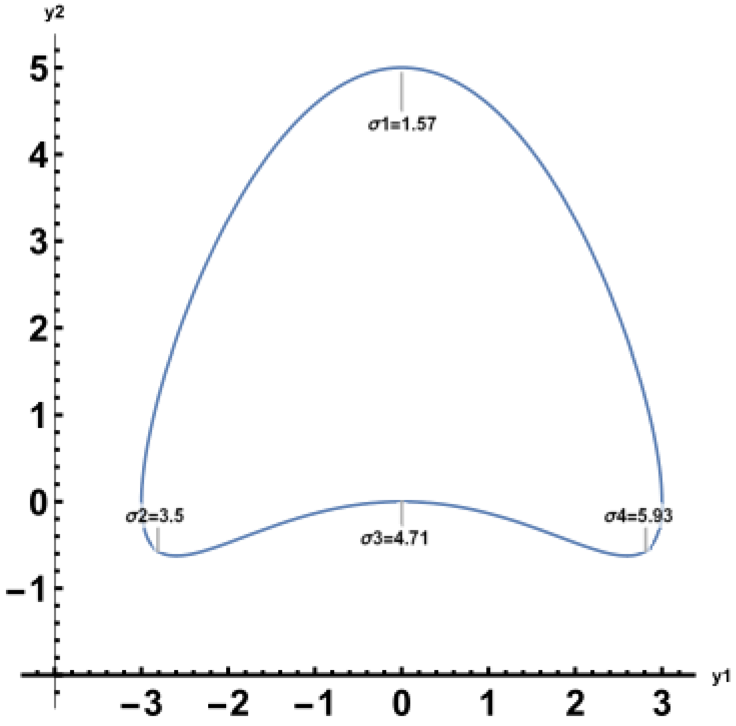
Free boundary condition: $\boldsymbol{\eta}(x,t) = 0$
the free boundary problem $x \in \Omega_t \in \mathbb{R}^2$

$$\boldsymbol{\eta}(\mathbf{x}, t) + D(\mathbf{x})|_{x \in \partial\Omega_t} = 0$$

Simple example: the basin with the parabolic bottom



More complicated shore:
it could be a lake or an island



bounded n and **large** m

$$D(x_1, x_2) = D(\rho) = a(R^2 - \rho^2)$$

Linear coastal waves $3D$:
Stokes, Ursell,
Dobrokhotov, Zhevandrov, Simonov, Isakov, Vukasinich, Merzon, Lyalinov,

Linear coastal on shallow water:
Anikin, Dobrokhotov, Minenkov, Nazaikinskii, Tsvetkova, Votyakova

Billiards with semi-rigid walls:
Anikin, Dobrokhotov, Minenkov, Nazaikinskii, Tsvetkova, Votyakova, Bolotin, Treshev

Nonlinear long waves: run-up and coastal waves
Carrier, Greenspane, Pelinovskii, Mazova, Dobrokhotov, Kalinichenko, Minenkov, Nazaikinskii, Tirozzi, Tsvetkova, Votyakova

Step 1) Naive linearization (trivial)

Step 2) Construction of asymptotic solutions to the linear spectral problem: billiards with semi-rigid walls, standard and nonstandard caustics, nonstandard Lioville tori, the modified Maslov canonical operator

Step 3) Approximate reconstruction of solutions of nonlinear problem via linear one: the modified Carrier-Greenspane transformation

S. Yu. Dobrokhotov, D. S. Minenkov, and V. E. Nazaikinskii, On Asymptotic Solutions of the Cauchy Problem for a Nonlinear System of Shallow Water Equations in a Basin with Gently Sloping Banks, Russian Journal of Mathematical Physics, Vol. 29, No. 1, 2022, pp. 28-36

Step 1. **Naive linearization** $N_t + \langle \nabla, D(x)\mathbf{U}\rangle = 0, \qquad \mathbf{U}_t + g\nabla N = 0$

this gives the 2-D wave equation equation

$$\frac{\partial^2 N}{\partial t^2} = \langle \nabla, gD(x)\nabla N\rangle, \qquad x \in \Omega$$

no boundary conditions:
we require the self-adjointness of the corresponding operator (energy limitation)

We consider the solutions in the form $N = Re(e^{i\omega t}\psi(x))$

$\Longrightarrow$ **Spectral problems in domains with boundary without boundary conditions** (O.A. Oleinik, E.V.Radkevich)

$$\mathcal{L}\psi = \lambda\psi, \qquad \mathcal{L} = -\langle \nabla, gD(x)\nabla\rangle, \qquad \lambda = \omega^2.$$

One has (linear) coastal waves (waves trapped by the cost)
if ψ is located near $\partial\Omega$

Defect: loss of splash process

Step 3) **Nonlinear shallow water equations**

$$\frac{\partial \boldsymbol{\eta}}{\partial t} + div((\boldsymbol{\eta} + D(x))\mathbf{u}) = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}, \nabla)\mathbf{u} + g\nabla\boldsymbol{\eta} = 0,$$

Linearization

$$N_t + \langle \nabla, D(x)\mathbf{U} \rangle = 0, \qquad \mathbf{U}_t + g\nabla N = 0.$$

The approximate solutions $(\boldsymbol{\eta}(x,t), \mathbf{u}(x,t))$ of a nonlinear system is given **by parametric formulas** (*one does not need to know almost anything !*)

$$x = y - N(y,t)\frac{\varrho(y)\nabla D(y)}{\|\nabla D(y)\|^2}, \qquad \boldsymbol{\eta} = N(y,t), \qquad \mathbf{u} = \mathbf{U}(y,t),$$

here $\varrho(x)$ is a cut off function, equal to one in some neighborhood of the boundary $\partial\Omega$ and zero outside the larger neighborhood.

The general formulas!

CARRIER-GREENSPAN TRANSFORM

THE LINEAR WAVE EQUATION for $N(\tau, y), U(\tau, y)$:

$$N_\tau + \frac{\partial}{\partial y}(\gamma^2 y U) = 0, \quad U_\tau + g N_y = 0, \qquad g = 1,\ \gamma = 1$$

CONSIDER the SYSTEM

$$x = y - N(\tau, y) + \frac{1}{2}U^2(\tau, y), \quad t = \tau + U(\tau, y)$$

small

Let it defines one-to-one map from $\{y \geq 0, \tau \in \mathbb{R}\}$ to the value area of the right hand side

THEN

$$\eta(t, x) = N(\tau, y) - \frac{1}{2}U^2(\tau, y), \quad v(t, x) = U(\tau, y)$$

are the solution to the **ORIGINAL NONLINEAR SYSTEM** in a **PARAMETRIC FORM**

$$\eta_t + \frac{\partial}{\partial x}[v(\eta + \gamma x)] = 0,\ v_t + v v_x + g\eta_x = 0$$

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Step 2) Billiards with semi-rigid walls, standard and nonstandard caustics, nonstandard Lioville tori, the modified Maslov canonical operator

Semiclassical asymptotics:

Example with radially symmetric parabolic bottom

The classical Hamiltonian with the degenerated metric

$$H = D(x)(p_1^2 + p_2^2) \quad D(x_1, x_2) = D(\rho) = a(R^2 - \rho^2)$$

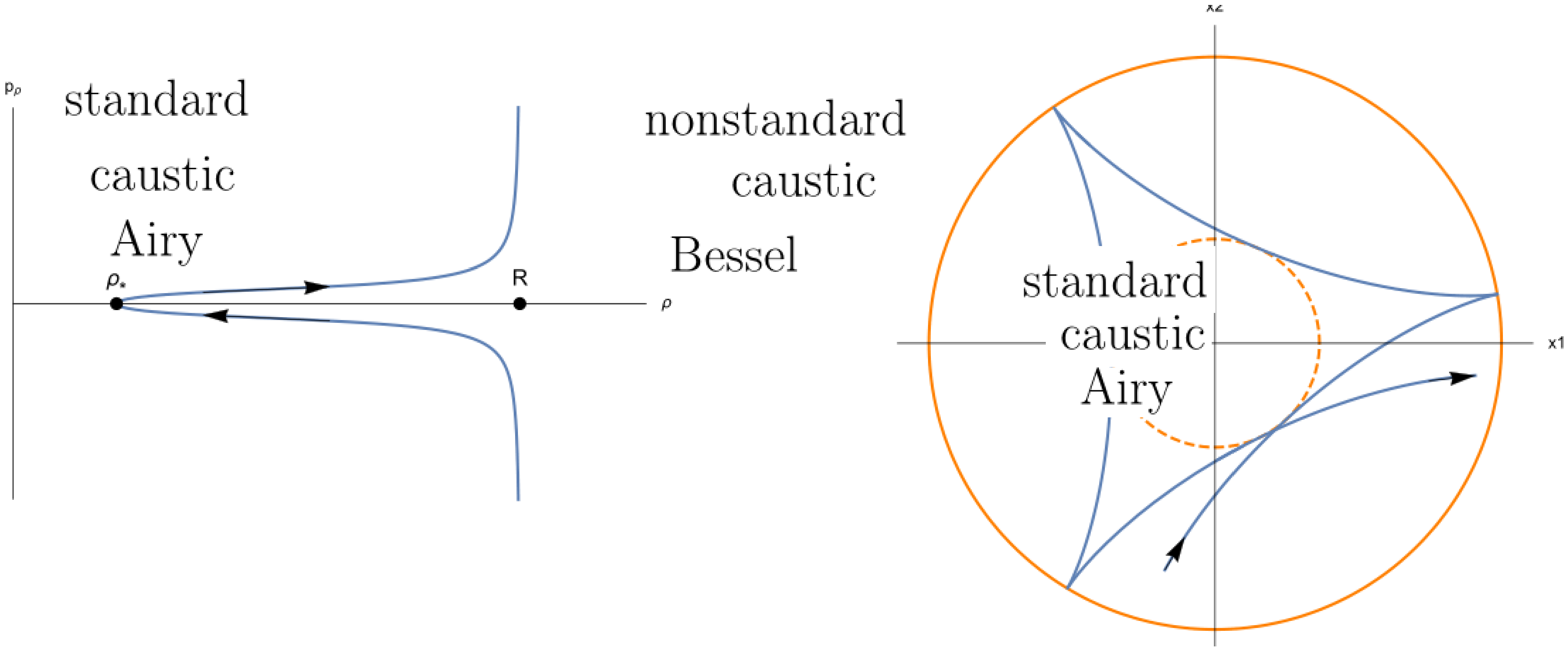
The 2D-invariant Lagrangian manifolds == nonstandard Liouville tori

$$\Lambda = \Lambda^2_{\omega,c} = \{H = a(R^2 - \rho^2)\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2}\right) = \omega^2, \ p_\phi = c\}$$

More general integrable case:

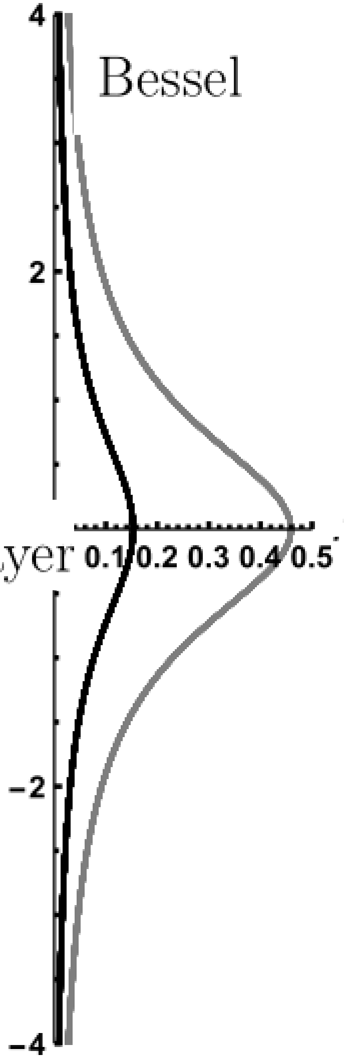
$$\Lambda = \Lambda^2_{\omega,c} = \{H(x,p) = \omega^2, \quad F(x,p) = c\}, \quad \{H, F\} = 0$$

the modified Maslov canonical operator+ uniformization



“Degenerated billiards with semirigid walls” and whispering gallery type asymptotic solutions

Bessel



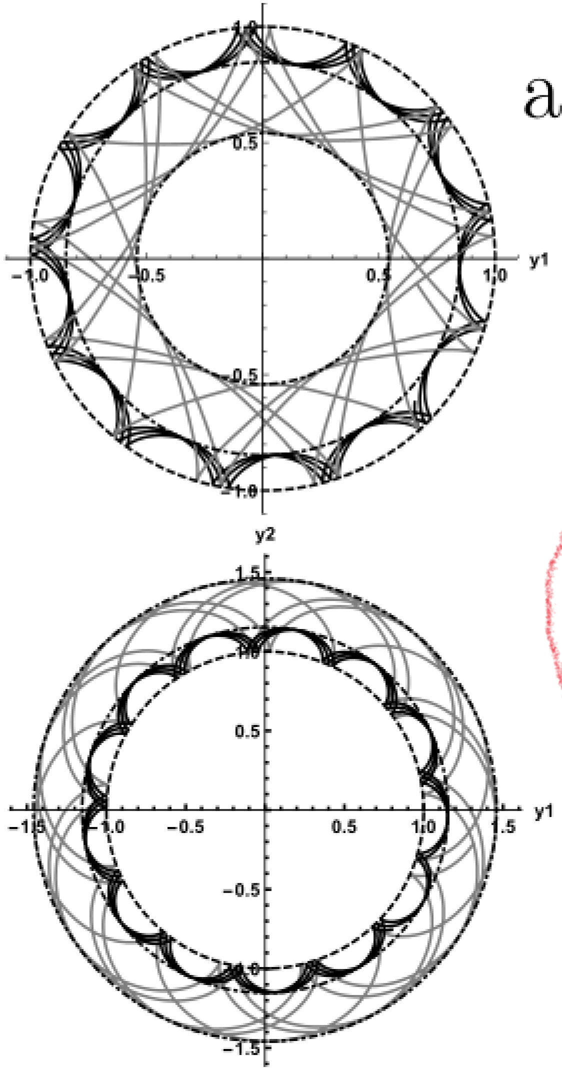
Airy

Boundary layer

+

Laguerre

(b)



(d)

analogues of **Stokes** and **Ursell** waves

Passage to nonintegrable case

(special spectral series)

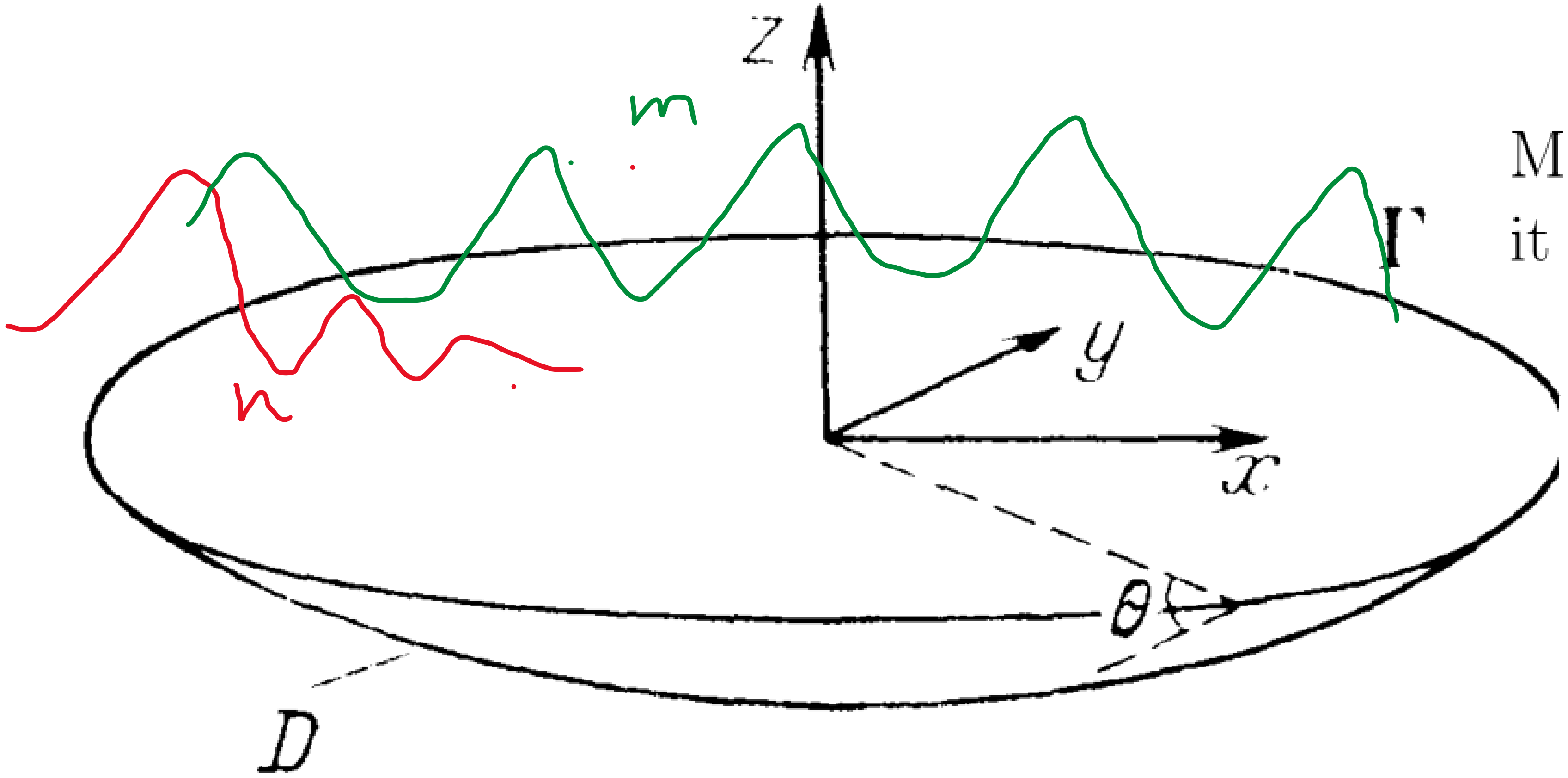
“Degenerated billiards with semi-rigid walls” (for water waves)

1.Partial (pseudo) differential equations
with fast and slow variables (operator valued symbols),

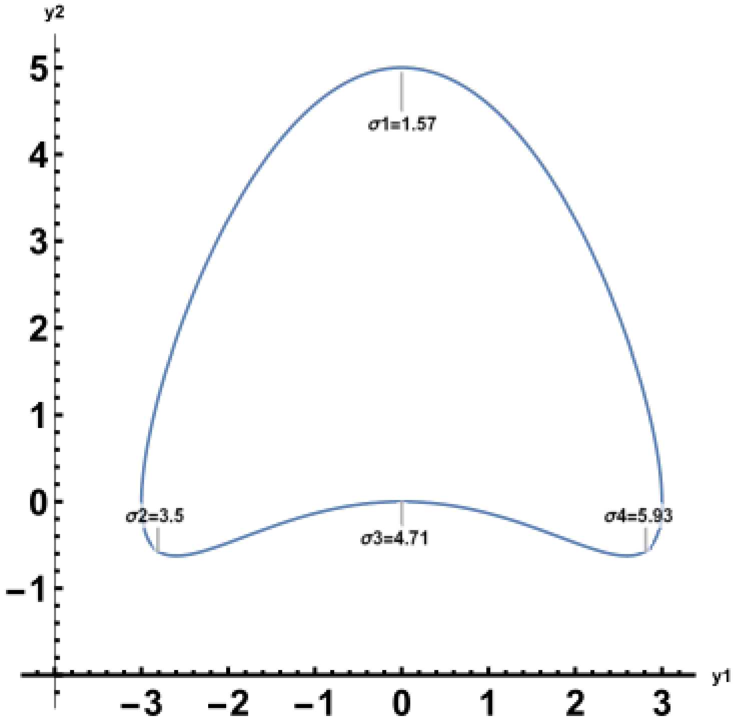
2. Born-Oppenheimer adiabatic approximation in the form of operator
separation of variables
based on the Heaviside-Feynman-Maslov calculus,

3. sequential dimensionality reduction and

4. effective approximate integration



More complicated shore:
it could be a lake or an island



bounded n and **large** m

$$D(x_1, x_2) = D(\rho) = a(R^2 - \rho^2)$$

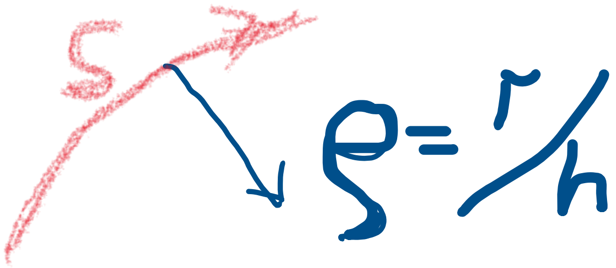
Appropriate coordinates in some small neighbourhood of the shore line Γ_0:

s be the natural parameter on Γ_0, $s \in [0, T]$ (the length of the curve Γ_0 measured from some fixed point) — **slow variable**

$\mathbf{n}(s)$ be the internal normal in Ω^0

$r = \mathbf{r}(x)$ be the distance from point x to the curve Γ_0, — **fast variable**

$s(x)$ determines the base of the normal to Γ_0 from point x



We have $x = \mathcal{Y}(r, s) = Y(s) + r\mathbf{n}(s)$

Denote the curvature of Γ_0: $\quad \kappa(s) = Y''(s), \quad \mathbf{n}(s) = \text{sign}(\kappa)|Y''(s)|$

We have $D = \gamma_0(s)r + \gamma_1(s)r^2 + O(r^3)$

Whispering gallery type waves

integrability is not required ! *convexity of Ω_0 is not required !*

oscillations alone the shore

$$\Psi_{nm}(r,s) = C_{nm}\sqrt{2p_n(s)\gamma_0(s)}\,e^{-\lambda_{nm}p_n(s)r}e^{i\lambda_{nm}\Phi_n(s)-i\Theta_n(s)}\mathbf{L}_n(2\lambda_{nm}p_n(s)r)$$

decreasing in the normal direction

$\mathbf{L}_n(z)$ is the Laguerre's polynomial

$$\omega_{nm}^2 = g\frac{2m\pi+\Theta_n(T)}{\Phi_n(T)}, \qquad \lambda_{nm}^{-1} = h_{nm} = \frac{\Phi_n(T)}{2m\pi+\Theta_n(T)}$$

Large m and bounded n

$$\Phi_n(s) = \frac{1}{2n+1}\int_0^s \frac{1}{\gamma_0(s)}ds, \qquad p_n(s) = \Phi_n'(s) \equiv \frac{1}{2n+1}\frac{1}{\gamma_0(s)},$$

$$\Theta_n(s) = \frac{3n^2+3n+\frac{1}{2}}{2n+1}\int_0^s \varkappa(s)ds + \frac{2n^2+2n+1}{2n+1}\int_0^s \frac{\gamma_1(s)}{\gamma_0(s)}ds$$

The reference evolution equation for the $n-$ th mode along the coastline

$$h^2\frac{\partial^2}{\partial t^2}\psi^\nu + \Big((2\nu+1)\gamma_0(s)\,| - ih\frac{\partial}{\partial s}| + hQ_\nu(s)\Big)\psi^\nu = 0, \qquad \nu = 0, 1, \ldots$$

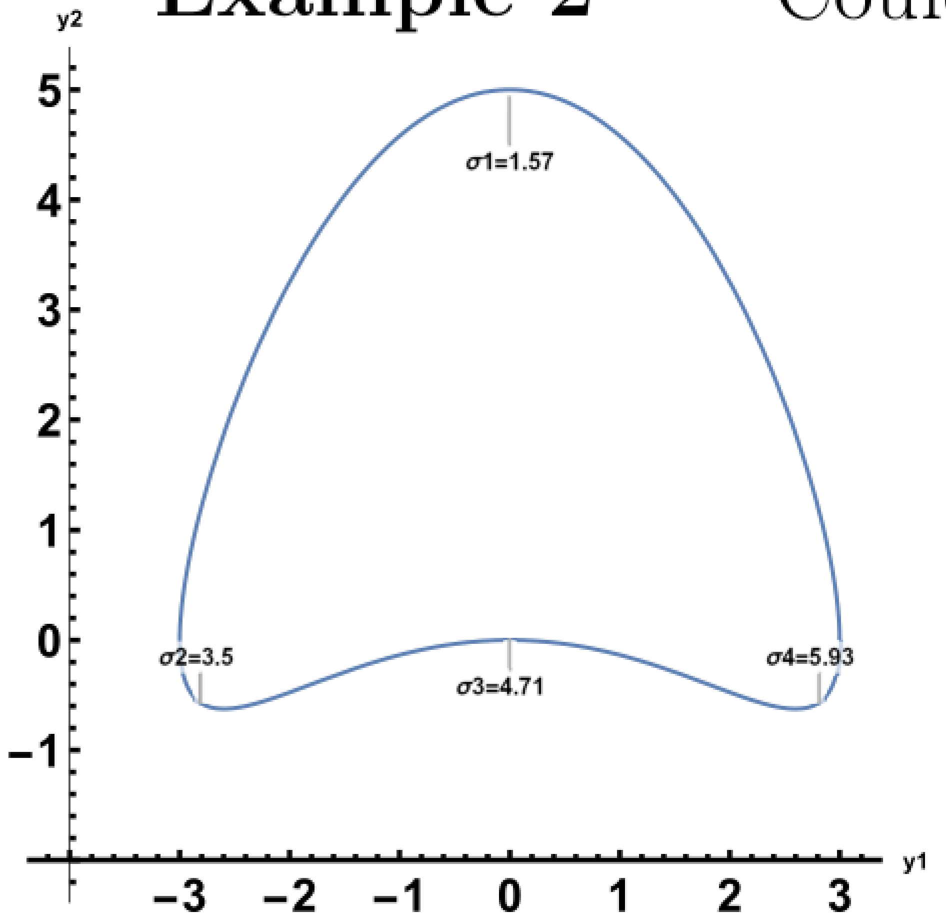
the dispersion relation looks like the deep water one

$$\omega^2 = \Big((2\nu+1)\gamma_0(s)|p| + hQ_\nu(s)\Big)$$

The arbitrary coordinate σ $\qquad y = \mathbf{Y}(\sigma) = (Y_1(\sigma), Y_2(\sigma))$

$$\{y_1 = Y_1(\sigma) = a\cos\sigma, \quad y_2 = Y_2(\sigma) = b\sin\sigma + q\sin^2\sigma, \quad \sigma \in [0, 2\pi]\}$$

Example 2 Could be the basin and could be the island

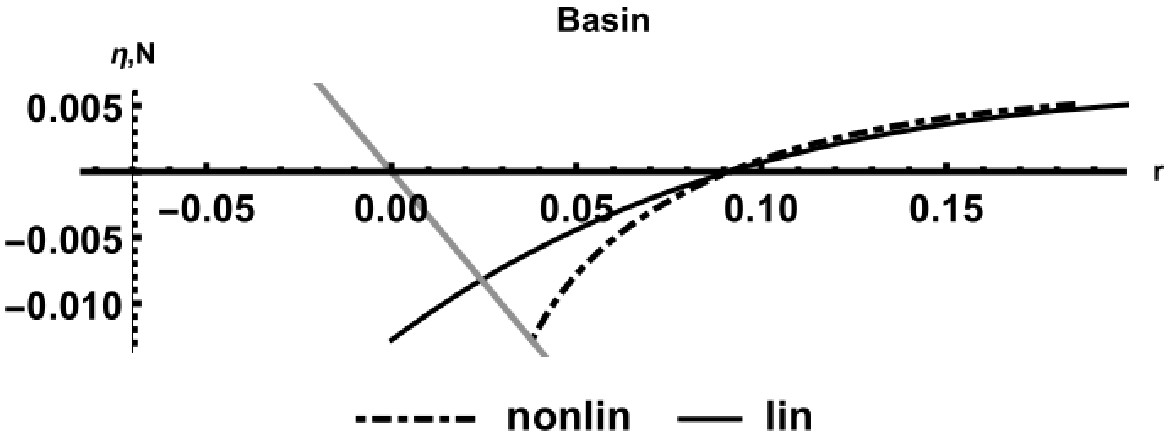


$$D(r,\sigma) = \gamma_0(\sigma)r + \gamma_1(\sigma)r^2 + O(r^3)$$

To construct the leading term

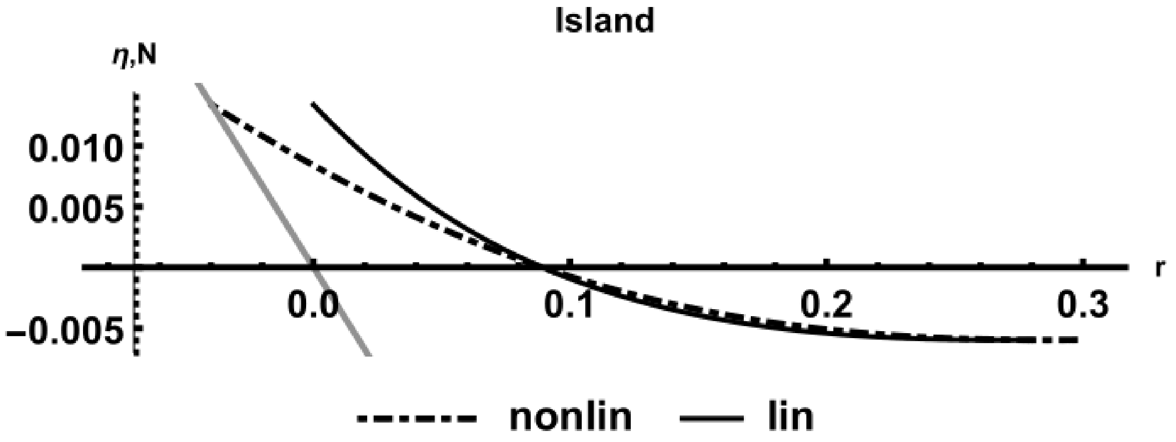
of asymptotic solutions describing the coastal

waves one needs 4 functions: $Y_{1,2}(\sigma), \gamma_{0,1}(\sigma)$



Basin
η,N
0.005
−0.005
−0.010
−0.05
0.00
0.05
0.10
0.15
r
nonlin
lin

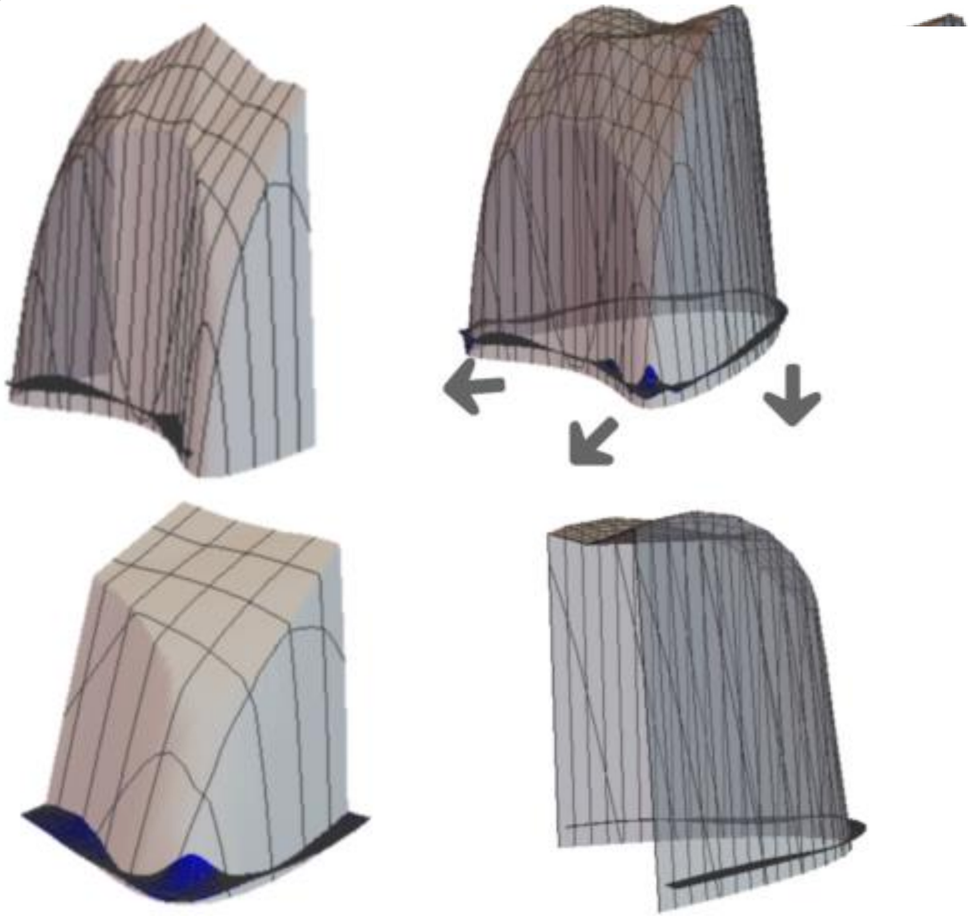
a)



Island
η,N
0.010
0.005
−0.005
0.0
0.1
0.2
0.3
r
nonlin
lin

(b)

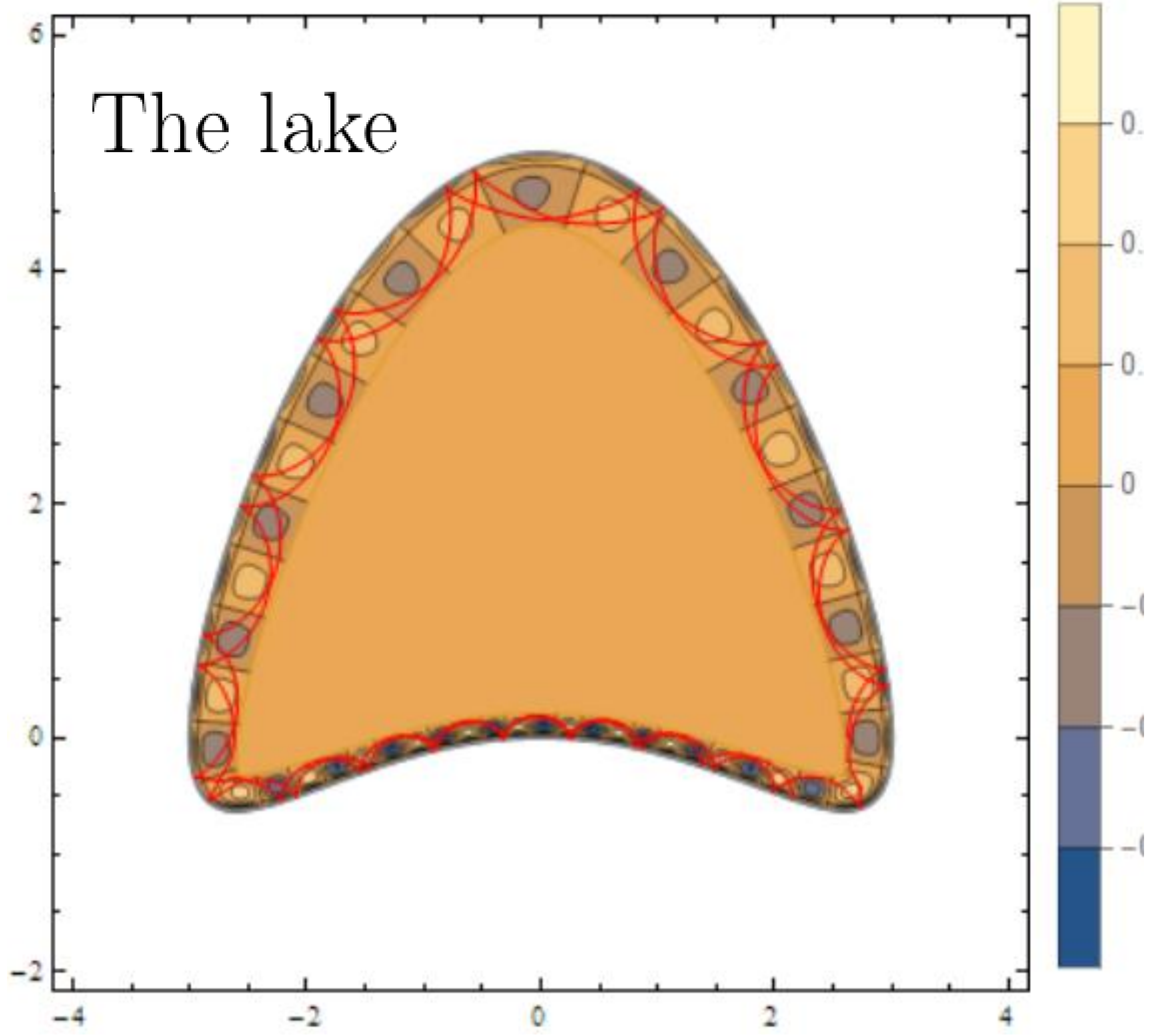
)

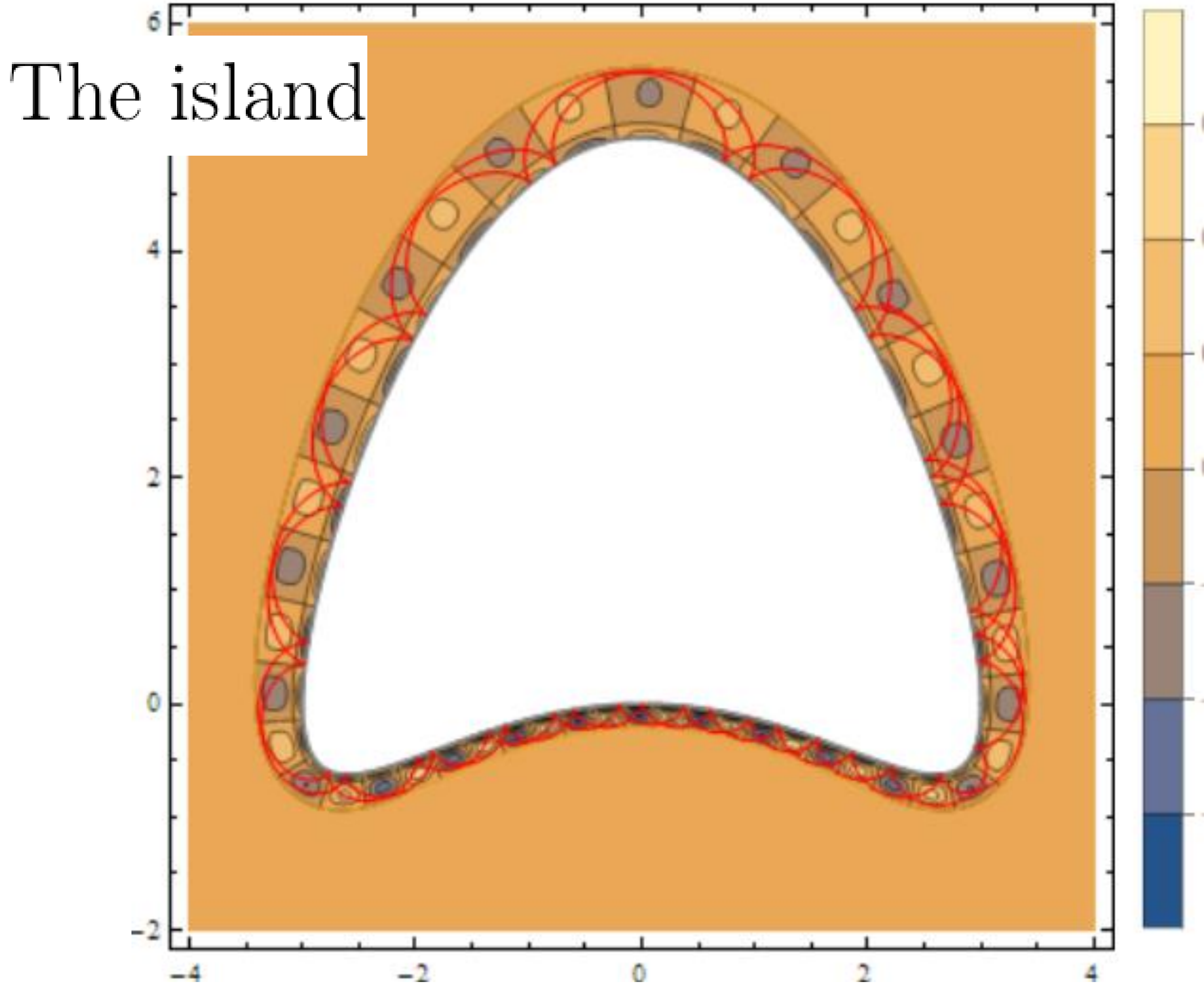


(d)

+ Using the last Bolotin-Treshev results and the Neishtadt averaging method in systems with a single fast variable

Relationship with Narrow billiard trajectoties





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S. Yu. Dobrokhotov, D. S. Minenkov, M. M. Votiakova, Classical and wave dynamics of nonlinear coastal waves, localized near gentle shores, Proceedings of the Steklov Institute of Mathematics, 2024, (to appear)

Спасибо за внимание!

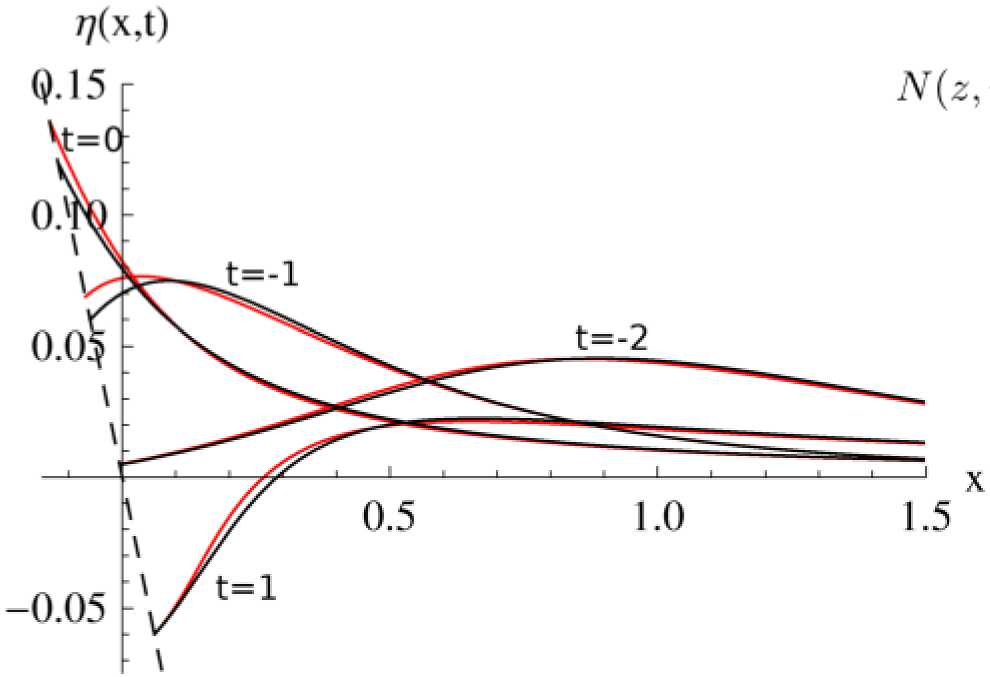
Reduced Carrier-Greenspan transformation (simplification):

$$D(y) = D(x) + \eta(x,t)\rho(x),\ \tau = t,\ N(y,\tau) = \eta(x,t),\ U(y,\tau) = u(x,t),$$

here $\rho(x)$ is a cut-off function, $\rho = 1$ near x^0 : $D(x) = 0$ and $\rho = 0$ outside of the neighborhood of the point x^0.

The main property: **the boundary becomes fixed**

Example: exact solitary solution



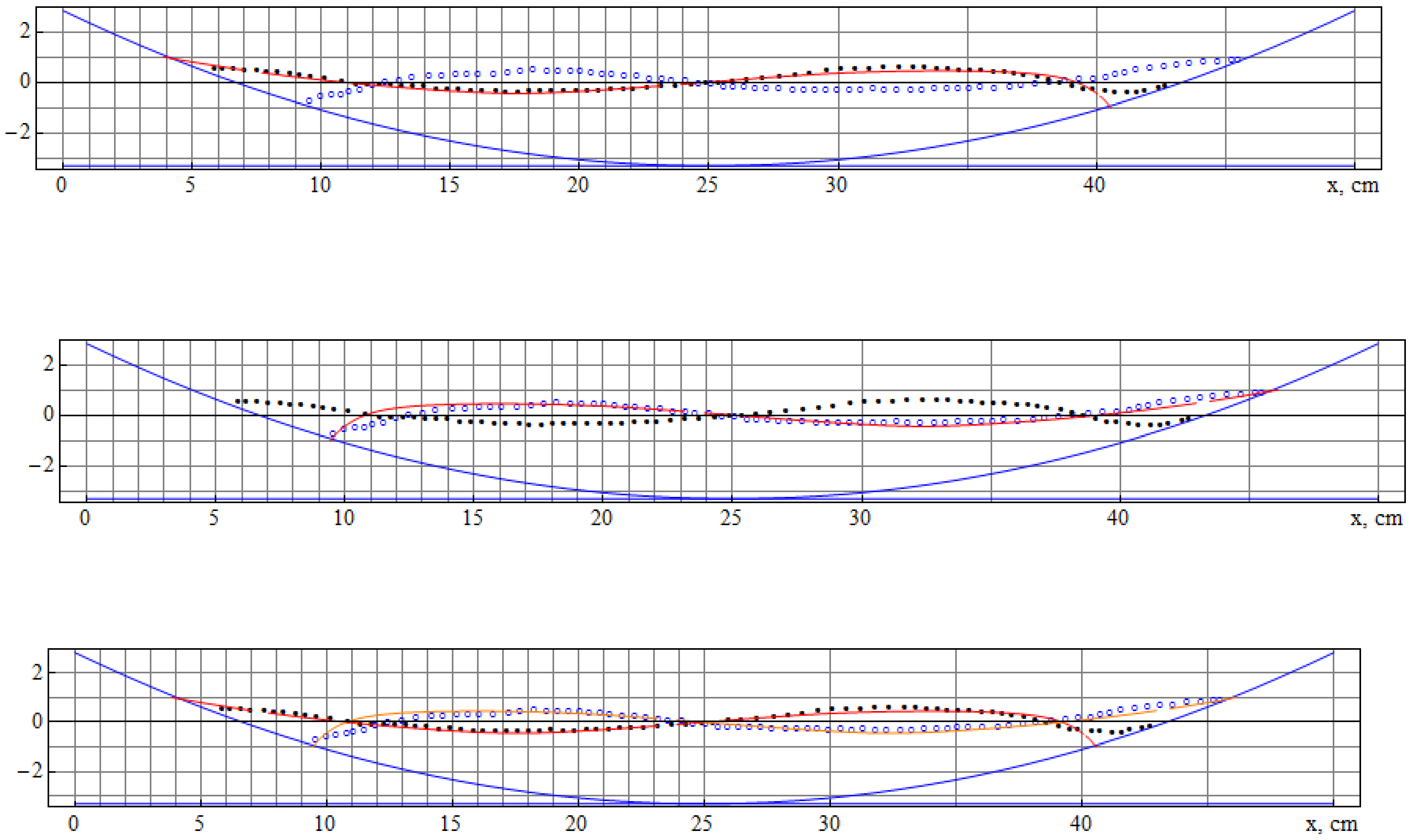
$$N(z,\tau) = \mathrm{Re}\,\frac{A(\tau+ib)}{(z-(\tau+ib)^2/4)^{3/2}},$$

$$U(z,\tau) = 2\,\mathrm{Re}\,\frac{A}{(z-(\tau+ib)^2/4)^{3/2}},$$

Experimental device (no tray)

Диапазон частот	0.1-5 Гц
Диапазон амплитуд	0.03-7.5 см
Угловые смещения	$< 8'$
Коэффициент нелинейных искажений	2%
Точность измерения периода	3×10^{-3} с
Допустимая масса сосуда с жидкостью	50 кг
Точность измерения пространственных характеристик волновых движений	1 мм

The electromechanical vibration stand provides vertical oscillation of the basin


2
0
−2
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5
10
15
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25
30
40
x, cm
2
0
−2
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5
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20
25
30
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x, cm
2
0
−2
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5
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30
40
x, cm

Adiabatic approximation in semiclassical case: $\rho, p_\rho \to -i\frac{\partial}{\partial\rho}$ are fast variables, $s, p_s \to -ih\frac{\partial}{\partial s}$ are slow variables

$$\hat{\mathcal{H}} = -\gamma_0(s)\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \gamma_0(s)r\frac{\partial^2}{\partial s^2} - \gamma_1(s)\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + r^2\big(2\gamma_0(s)\varkappa(s) + \gamma_1(s)\big)\frac{\partial^2}{\partial s^2} + r\gamma_0(s)\varkappa(s)\frac{\partial}{\partial r} - ir\gamma_0'(s)\frac{\partial}{\partial s} + O(r^3)$$

$$\hat{\mathcal{H}}\Psi := \mathcal{H}(\rho, \overset{2}{s}, \frac{\partial}{\partial\rho}, \overset{1}{\hat{p}}, h)\Psi = \lambda\Psi, \quad \mathcal{H} = \mathcal{H}_0 + h\mathcal{H}_1 + O(h^2), \quad \mathcal{H}_0(\rho, \frac{\partial}{\partial\rho}, s, p) = -\gamma_0(s)\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \gamma_0(s)\rho p^2 - 1,$$

$$\mathcal{H}_1(\rho, \frac{\partial}{\partial\rho}, s, p) = -\gamma_1(s)\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho} + \rho\gamma_0(s)\varkappa(s)\frac{\partial}{\partial\rho} + \rho^2\big(2\gamma_0(s)\varkappa(s) + \gamma_1(s)\big)p^2 - i\rho\gamma_0'(s)p.$$

$$\Psi(\rho, s, h) = \hat{\chi}_0\psi(s, h), \quad \hat{\chi}_0 = \chi(\rho, \overset{2}{s}, \overset{1}{\hat{p}}, h), \quad \chi_0(\rho, s, p, h) = \sqrt{2|p|}e^{-|p|\rho}\mathbf{L}_n(2|p|\rho).$$

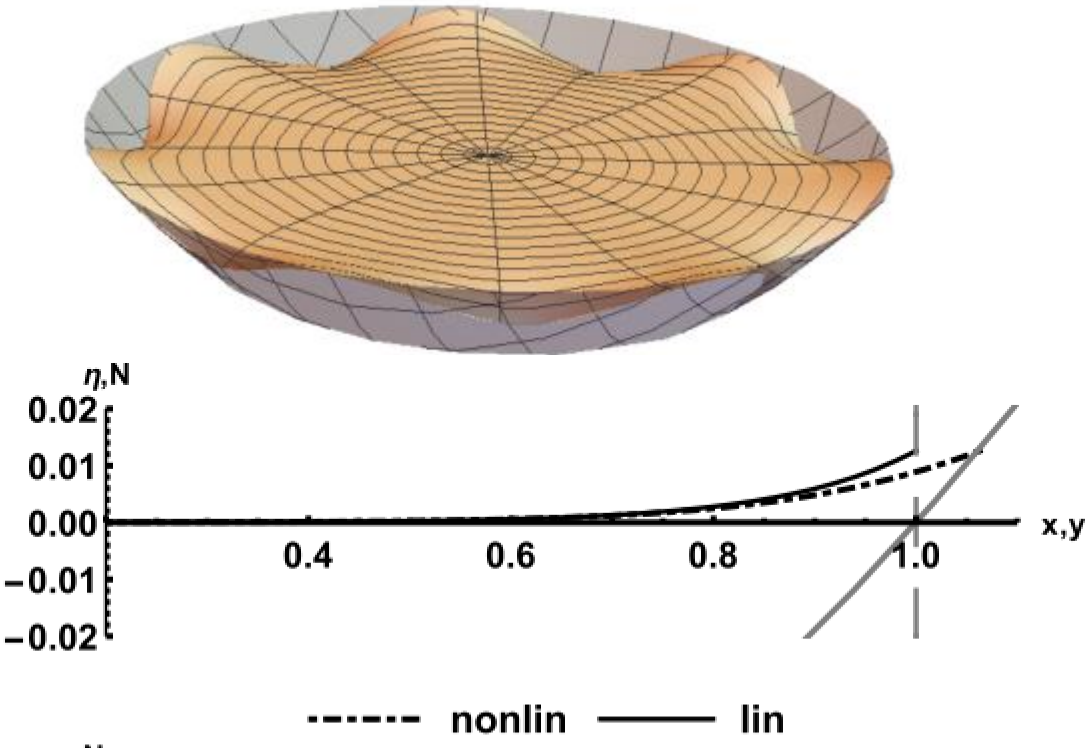
Adiabatic approximation in classical case: r, ξ are fast variables, s, p are slow variables

$$\mathcal{H}(r, \xi, s, p) = \mathcal{H}_0(r, \xi, s, p) + \mathcal{H}_1(r, \xi, s, p) + O(r^3),$$

$$\mathcal{H}_0(r, \xi, s, p) = \gamma_0(s)r\,(\xi^2 + p^2), \quad \mathcal{H}_1 = r^2\Big(\gamma_1(s)\,(\xi^2 + p^2) + 2\gamma_0(s)\varkappa(s)p^2\Big)$$

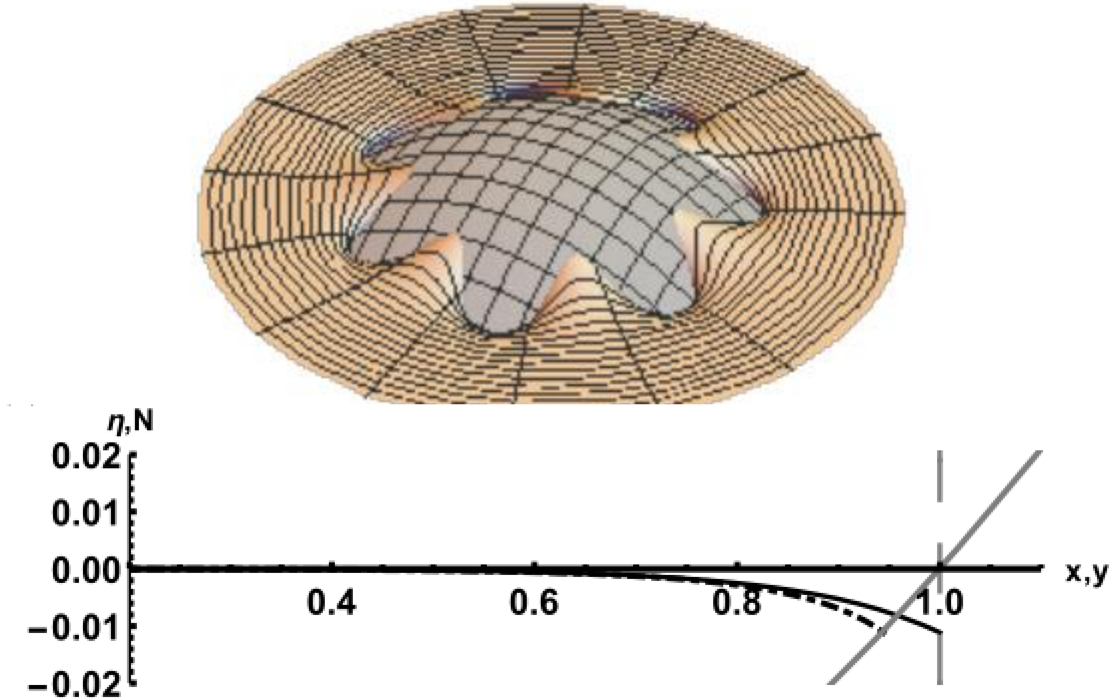
Example 1: radially symmetric parabolic bottom

the parabolic basin

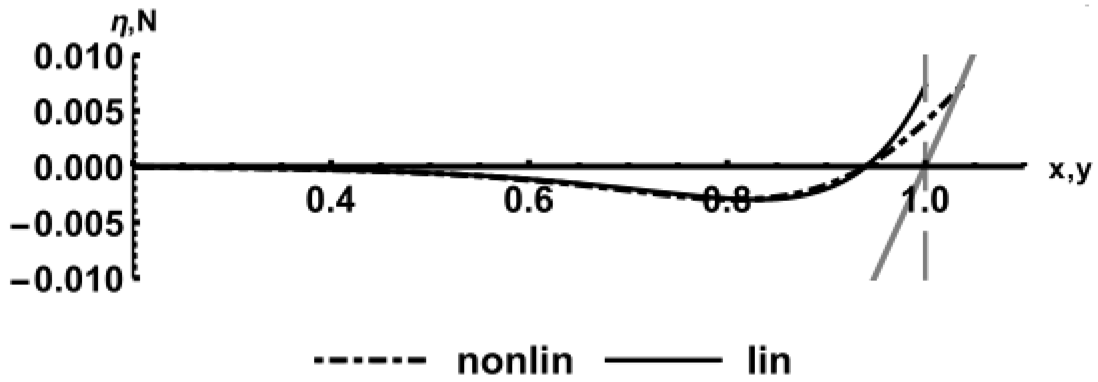


(a)

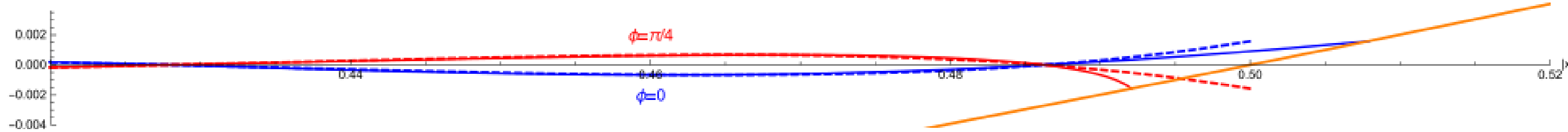
the parabolic island



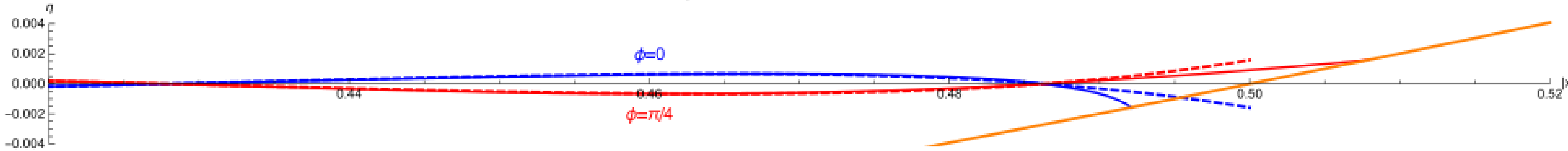
(b)



Linear and Nonlinear asymptotics near the shoreline



$t = 0$, масштаб осей $1:1$



$t = 0.7$, масштаб осей $1:1$

S. Yu. Dobrokhotov, V. E. Nazaikinskii, A.V. Tsvetkova, Nonlinear effects and splash of coastal waves in the framework of shallow water theory, Proc. Steklov Inst. Math., (2023)

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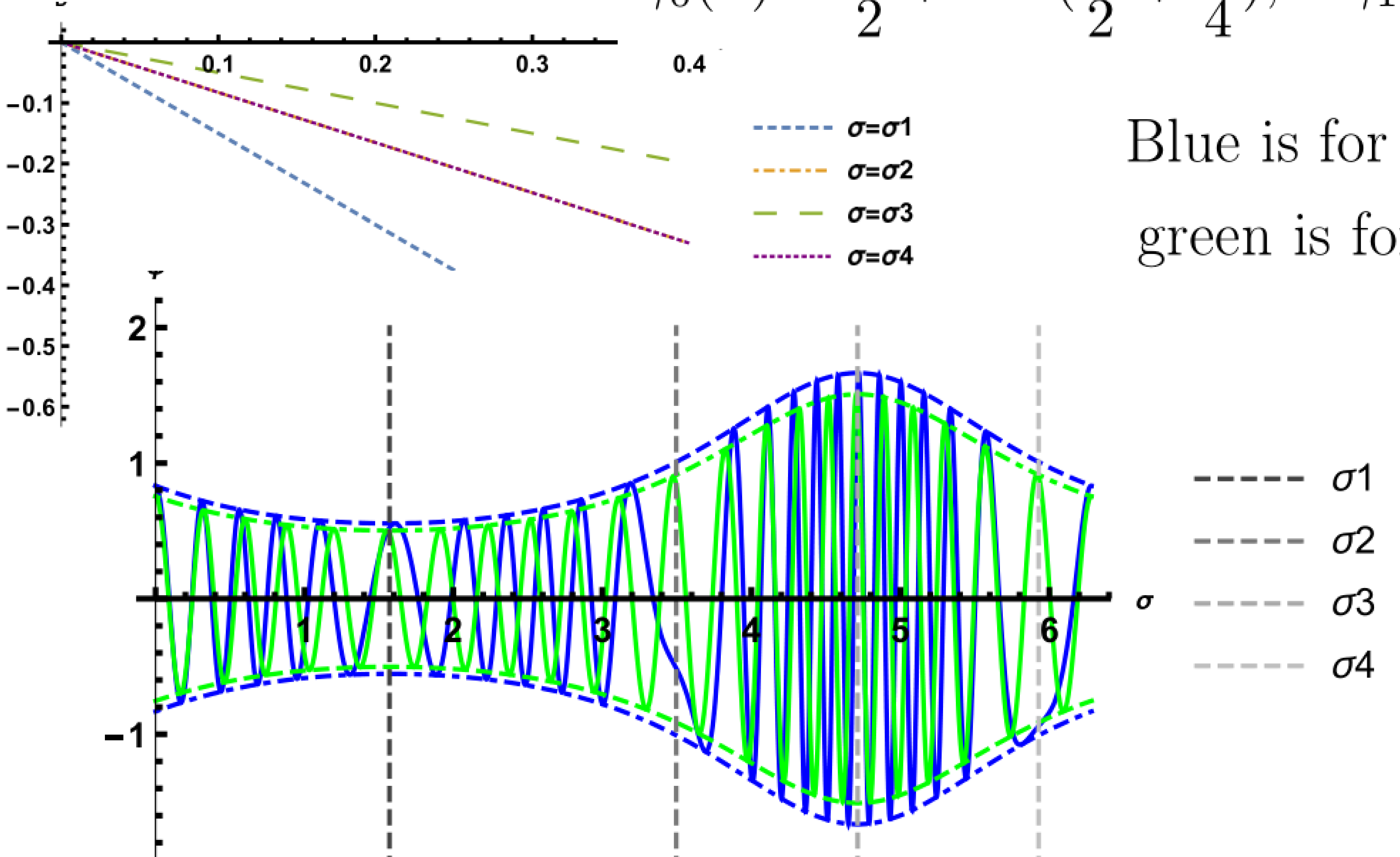
The linear wave dependence on the bottom angle

$$\gamma_0(\sigma) = \frac{1}{2} + \sin^2\left(\frac{\sigma}{2} + \frac{\pi}{4}\right), \quad \gamma_1(\sigma) = 0.$$



Blue is for the lake and green is for the island